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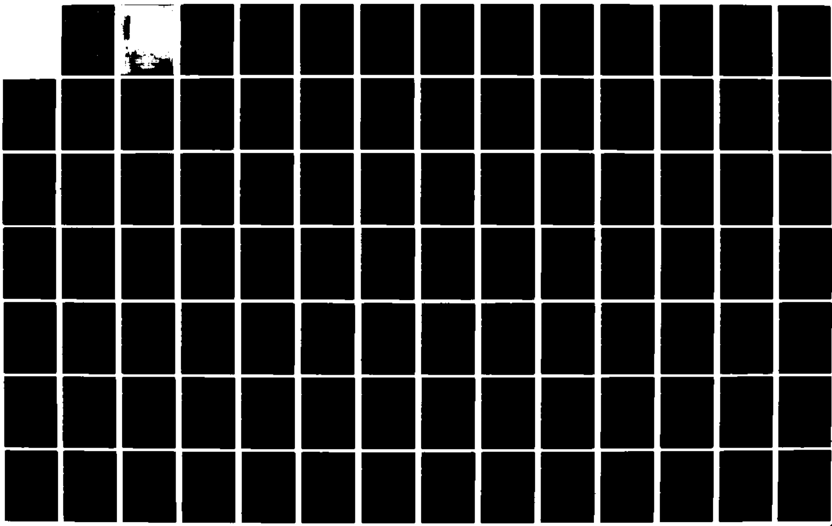

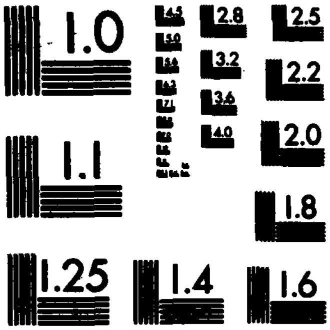

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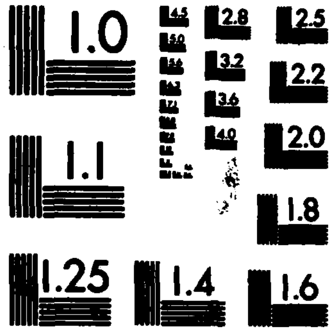

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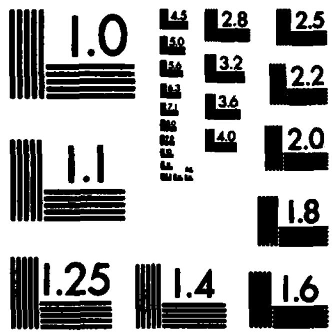

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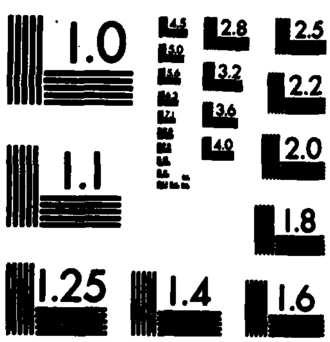

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space (Ω, β, p), where X is a non-negative random variable and $\xi(\cdot, \omega)$ is a function defined for $0 \le t \le X(\omega)$ taking its values in some abstract space X. Let F be the distribution of X; assume $F(0) < 1$ and $F(\infty) = 1$.

Let (Ω_i, β_i, p_i), $i \ge 1$, be independent copies of the probability space and let P be the product measure defined on $\prod_{i=1}^{\infty} \beta_i$. Suppose θ_1, θ_2, ... is a sequence of tours where each $\theta_i = (X(\omega_i), \xi(t, \omega_i))$ has domain Ω_i and is chosen in accordance with the probability measure p_i. Writing $X_i = X(\omega_i)$, we see that $\{X_i\}$ is a renewal process. We can construct a cumulative process as follows

$$W(t) = \xi(t, \omega_1), \; t \le X_1$$

TRANSIENT CUMULATIVE PROCESSES

by

Emily Stough Murphree

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EMILY STOUGH MURPHREE. Transient Cumulative Processes (Under the direction of WALTER L. SMITH.)

Suppose (X_1, Y_1) has the improper distribution $G(x,y)$ and X_1 is a positive lifetime whose distribution F is such that $F(\infty) = \omega < 1$. Independent vectors (X_1, Y_1), (X_2, Y_2), $\cdots$ having distribution G are sampled until an infinite lifetime is chosen, when the renewal process is said to "die." Let $N(t)$ be the number of lifetimes observed by time t and let $A(t) = 1$ if $X_1 < \infty, \cdots, X_{N(t)+1} < \infty$ and 0 otherwise. When $A(t) = 1$, define the Type B cumulative process

$$W(t) = \sum_{j=1}^{N(t)} Y_j \ .$$

To ensure $W(t)$ is defined, expectations and probabilities of interest are conditioned upon $\{A(t)=1\}$. Two sharply delineated situations arise. Although many standard renewal theoretic results have direct analogs in the first, they do not in the second.

In the first case, there is some $\sigma > 0$ making

$$\int_0^\infty \int_{-\infty}^\infty e^{\sigma x} G(dx,dy) = 1$$

and a new d.f.

$$\tilde{G}(x,y) = \int_0^x \int_{-\infty}^y e^{\sigma u} G(du,dv)$$

is defined. Under general conditions on the product moments of $\tilde{G}$, $E[W(t)^k | A(t)=1]$ is a $k\underline{th}$ degree polynomial in t plus $R_k(t)$, a term converging to zero at a rate depending upon assumptions about $\tilde{G}$. These same conditions ensure that the $k\underline{th}$ conditional cumulant of $W(t)$ is $A_k t + B_k + R_k(t)$.

Results about the conditional asymptotic normality of $W(t)$ are also obtained.

In the second case, the tail of F is a function of slow growth; a technical assumption implies

$$\lim_{t\to\infty} P\{N(t)=n|A(t)=1\} > 0$$

and

$$\lim_{t\to\infty} E[N(t)^k|A(t)=1] = \alpha_k < \infty.$$

Several examples of transient cumulative processes are considered and the assumption that each X_i has the same defect $1-\omega$ is relaxed.

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ACKNOWLEDGEMENTS

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Finally, I appreciate Linda Tingen's careful typing of the manuscript.

NOTATION

$$\mathcal{L}\{J(t)\} = J^{*}(s) = \int_0^{\infty} e^{-st}\, dJ(t)$$

$$\mathcal{L}^{0}\{J(t)\} = J^{0}(s) = \int_0^{\infty} e^{-st}\, J(t)dt$$

Note: All functions J are assumed to be zero on $(-\infty,0)$.

$F(t)*G(t) = \int_0^t F(t-\tau)dG(\tau)$, the Stieltjes convolution of F and G.

$F^{(n)}(t) = \int_0^t F(t-\tau)dF^{(n-1)}(\tau)$, the n-fold Stieltjes convolution of F with itself. $(n = 2, 3,\cdots)$

$U(t) = P\{0 \le t\}$

$$\left.\begin{array}{l} \tilde{F}(x) = \int_0^x e^{\sigma y}\, dF(y) \\ \tilde{G}(x,y) = \int_0^x \int_{-\infty}^{y} e^{\sigma u}\, G(du,dv) \end{array}\right\} \text{whenever there is a } \sigma \text{ making } \tilde{F} \text{ and } \tilde{G} \text{ proper distributions}$$

Any expression of the form $\tilde{\mu}_r$, $\tilde{\kappa}_s$, $\tilde{\mu}_{rs}$, $\tilde{\gamma}$, etc. has its usual meaning with the understanding that expectations have been taken with respect to $\tilde{F}$ and $\tilde{G}$.

$\mathcal{C}$ is the class of distributions F such that $F^{(k)}$ has an absolutely continuous component for some finite k.

L_1 is the class of Lebesgue integrable functions.

S is the class of distributions G on $(0,\infty)$ such that $G(0+) = 0$, $G(x) < 1$ for $0 < x < \infty$, and

$$\lim_{x\to\infty} \frac{1-G^{(2)}(x)}{1-G(x)} = 2.$$

CHAPTER 1

INTRODUCTION

1.1 RENEWAL THEORY

A renewal process is a sequence of independent, identically distributed positive random variables $X_1, X_2, \ldots$. Let X_i have distribution function F; assume $F(0) < 1$ and write $\mu_k = EX_i^k \leq \infty$. Typically, the X's are interpreted as waiting times for some prescribed event to occur. If the event occurs at time t, we begin waiting all over again for the next occurrence, and the process beginning at t is a probabilistic replica of the process which began at time 0.

The X's could be lifetimes of pieces of equipment and the recurrent event the failure of the equipment. At time 0, the first piece is installed; it fails at time X_1 and is instantly replaced. X_n is the lifetime of the nth piece of equipment while the partial sum $S_n = \sum_{i=1}^{n} X_i$ is the time of the nth renewal.

Let N(t) be the integer k such that $S_k \leq t < S_{k+1}$. N(t) is the number of events by time t. A great deal of research in renewal theory has centered around the renewal function $H(t) \equiv EN(t)$, for knowledge of the properties of H allows one to answer most questions arising about a renewal process. H(t) may be written

$$H(t) = EN(t) = E \sum_{1}^{\infty} \chi(S_n \leq t) = \sum_{n=1}^{\infty} F^{(n)}(t), \qquad (1.1.1)$$

where $F^{(n)}(t)$ denotes the n-fold convolution of F with itself. H satisfies the integral equation of renewal theory

$$H(t) = F(t) + \int_0^t H(t-\tau)dF(\tau). \tag{1.1.2}$$

In the usual literature, F is a proper distribution. That is, $\lim_{t\to\infty} F(t) = 1$. In this case the asymptotic behavior of N(t) and H(t) is well known.

One of the earliest results concerning H(t) is the Elementary Renewal Theorem (Feller, (1941)), which states

$$\frac{1}{t} H(t) \to \frac{1}{\mu_1} , \tag{1.1.3}$$

where here and elsewhere we interpret $\frac{1}{\mu_1}$ as 0 when $\mu_1 = \infty$.

The waiting times $\{X_i\}$ are d-lattice random variables if

$$\sum_{n=1}^{\infty} P\{X_i = nd\} = 1 \tag{1.1.4}$$

and d is the largest number for which (1.1.4) holds. If the X's are lattice random variables, the renewal process is *discrete*; if not, it is *continuous*. In all the work which follows, we shall assume the process is continuous. There is usually a parallel theorem for discrete processes to any theorem about continuous processes.

Blackwell's Theorem (1948) is an important generalization of the Elementary Renewal Theorem for continuous processes. It states that for any fixed $\alpha > 0$

$$H(t+\alpha) - H(t) \to \frac{\alpha}{\mu_1} \quad \text{as} \quad t \to \infty. \tag{1.1.5}$$

Blackwell's Theorem was extended by the Key Renewal Theorem, proved by W. L. Smith in 1954 and stated in its most general form in 1961.

Key Renewal Theorem (Smith 1961)

If $F(x)$ is a non-lattice distribution and $Q(x)$ is Riemann integrable in every finite interval and satisfies

$$\sum_{n=0}^{\infty} \max_{n \le x \le n+1} |Q(x)| < \infty, \text{ then} \tag{1.1.6}$$

$$\int_0^x Q(x-\tau)dH(\tau) \to \frac{1}{\mu_1} \int_0^\infty Q(\tau)d\tau \quad \text{as } x \to \infty . \tag{1.1.7}$$

Feller calls functions satisfying the conditions of this theorem "directly Riemann integrable."

The Key Renewal Theorem does not require the finiteness of any moments of F; however, by assuming $\mu_2 < \infty$, Smith (1954) proved

$$H(t) - \frac{t}{\mu_1} = \frac{\mu_2}{2\mu_1^2} - 1 + o(1) \text{ as } t \to \infty \tag{1.1.8}$$

by applying the theorem to a judiciously chosen function Q.

The theorem was instrumental in his proof that

$$\mathrm{Var}(N(t)) = \frac{\mu_2 - \mu_1^2}{\mu_1^3} t + o(t) \text{ as } t \to \infty \tag{1.1.9}$$

when $\mu_2 < \infty$.

The fact that the first two cumulants of $N(t)$ are asymptotically linear functions in t led Smith to investigate higher order cumulants. Let C be the class of distribution functions F such that for some finite k, $F^{(k)}$ has an absolutely continuous component.

<u>Theorem</u> (Smith 1959)

If $F \in C$ and $\mu_{n+p+1} < \infty$, $p \geq 0$, then there exist constants a_n and b_n such that the <u>nth</u> cumulant of $N(t)$ is given by

$$a_n t + b_n + \frac{\lambda(t)}{(1+t)^p}, \tag{1.1.10}$$

where $\lambda(t)$ is of bounded total variation, is $o(1)$ as $t \to \infty$, satisfies the condition

$$\lambda(t) - \lambda(t-\alpha) = o(t^{-1}) \text{ as } t \to \infty \tag{1.1.11}$$

for every fixed $\alpha > 0$, and when $p \geq 1$ has the additional property that $\frac{\lambda(t)}{1+t}$ belongs to the class L_1.

In addition to $N(t)$, there are two other random variables associated with time which require our attention. $\zeta_t = S_{N(t)+1} - t$ is called the forward delay and represents the waiting time from t until the next event. $\eta_t = t - S_{N(t)}$ is the backward delay, the elapsed time since the last event. $\eta_t + \zeta_t = X_{N(t)+1}$, the lifetime spanning time t.

When $\mu_1 < \infty$, limiting distributions for η_t and ζ_t can be calculated via the Key Renewal Theorem. One can show that

$$P\{\zeta_t \leq x\} = F(t+x) - \int_0^t [1-F(t+x-\tau)]dH(\tau) \text{ and hence}$$

$$\lim_{t\to\infty} P\{\zeta_t \leq x\} = 1 - \frac{1}{\mu_1}\int_x^\infty [1-F(\tau)]d\tau$$

$$= \frac{1}{\mu_1}\int_0^x [1-F(\tau)]d\tau \equiv K(x). \tag{1.1.12}$$

Similarly, if $\mu_1 < \infty$,

$$\lim_{t\to\infty} Ee^{-\theta\zeta_t} = \frac{1}{\theta\mu_1} [1-F^*(\theta)] = K^*(\theta). \tag{1.1.13}$$

For the backward delay, we find

$$P\{\eta_t \le x\} = U(x-t)[1-F(t)] + \int_{t-x}^{t} [1-F(t-\tau)]dH(\tau)$$

where $U(t) = P\{0 \le t\}$. Again assuming $\mu_1 < \infty$,

$$\lim_{t\to\infty} P\{\eta_t \le x\} = \frac{1}{\mu_1} \int_0^x [1-F(\tau)]d\tau = K(x). \tag{1.1.14}$$

The forward and backward delays have the same asymptotic distributions.

Just as limiting distributions for the forward and backward delays can be derived, there are asymptotic distributional results about $N(t)$. Using the relationship $P\{N(t) \ge n\} = P\{S_n \le t\}$, Feller (1941) proved that if $\mu_2 < \infty$ and $\sigma^2 = \mu_2 - \mu_1^2$, then

$$P\{N(t) > \frac{t}{\mu_1} - \frac{\alpha\sigma}{\mu_1}\sqrt{\frac{t}{\mu_1}}\} \to \Phi(\alpha) \text{ as } t \to \infty. \tag{1.1.15}$$

Feller's proof dealt specifically with discrete renewal processes, but his argument can be extended to cover the continuous case.

1.2 CUMULATIVE PROCESSES

Cumulative processes are a natural extension of renewal processes. A cumulative process is a stochastic process built up from processes of random length called *tours* in a special way. A random tour consists of an ordered pair $(X(\omega), \xi(t,\omega))$ defined on a probability

space (Ω,β,p), where X is a non-negative random variable and $\xi(\cdot,\omega)$ is a function defined for $0 \le t \le X(\omega)$ taking its values in some abstract space $\mathcal{X}$. Let F be the distribution of X; assume $F(0) < 1$ and $F(\infty) = 1$.

Let (Ω_i,β_i,p_i), $i \ge 1$, be independent copies of the probability space and let P be the product measure defined on $\prod_{i=1}^{\infty} \beta_i$. Suppose θ_1, θ_2, ... is a sequence of tours where each $\theta_i = (X(\omega_i),\xi(t,\omega_i))$ has domain Ω_i and is chosen in accordance with the probability measure p_i. Writing $X_i = X(\omega_i)$, we see that $\{X_i\}$ is a renewal process. We can construct a cumulative process as follows

$$W(t) = \xi(t, \omega_1), \; t \le X_1$$

$$\sum_{i=1}^{N(t)} \xi(X_i,\omega_i) + \xi(t-S_{N(t)},\omega_{N(t)+1}), \; S_{N(t)}<t\le S_{N(t)+1}.$$

We require that W(t) be of bounded variation in every finite t interval with probability one and that the random variables

$$Y_i^* = \int_{S_{i-1}}^{S_i} |dW(t)| \quad \text{also be iid.}$$

Let $Y_i = \xi(X_i,\omega_i)$; then $W(t) = \sum_{i=1}^{N(t)} Y_i + \xi(t-S_{N(t)}, \omega_{N(t)+1})$.

At time t the process is therefore the sum of N(t) iid random variables plus an extra piece depending upon the tour in progress at time t.

Write $\kappa_r = EY_i^{\,r}$ and $\kappa_r^* = EY_i^{*r}$ when these moments exist. Let $\sigma_x^2 = \mathrm{Var}(X_i)$, $\sigma_y^2 = \mathrm{Var}(Y_i)$, and $\rho_{xy} = \mathrm{Cov}(X_i, Y_i)$. Smith (1955) has shown that

$$\lim_{t\to\infty} \frac{1}{t} W(t) = \frac{\kappa_1}{\mu_1} \text{ a.s. if } \mu_1 < \infty,\ \kappa_1^* < \infty \tag{1.2.1}$$

$$EW(t) = \frac{\kappa_1}{\mu_1} t + o(t) \text{ if } \mu_1 < \infty,\ \kappa_1^* < \infty\ . \tag{1.2.2}$$

Define $\gamma = \sigma_y^2 - 2\rho_{xy}\sigma_x\sigma_y(\frac{\kappa_1}{\mu_1}) + \sigma_x^2(\frac{\kappa_1}{\mu_1})^2$. Then

$$\mathrm{Var}(W(t)) = \frac{t}{\mu_1}\gamma + o(t) \text{ if } \mu_2 < \infty, \kappa_2^* < \infty \tag{1.2.3}$$

$$\lim_{t\to\infty} P\{ \frac{W(t)-\kappa_1 N(t)}{\sigma_y \sqrt{\frac{t}{\mu_1}}} \le \alpha\} = \Phi(\alpha) \text{ if } \kappa_2^* < \infty,\ \mu_1 < \infty \tag{1.2.4}$$

and

$$\lim_{t\to\infty} P\{ \frac{W(t) - \frac{\kappa_1 t}{\mu_1}}{\sqrt{\frac{\gamma t}{\mu_1}}} \le \alpha\} = \Phi(\alpha) \text{ if } \kappa_2^* < \infty,\ \mu_2 < \infty. \tag{1.2.5}$$

Equations (1.2.2) and (1.2.3) suggest that the cumulants of $W(t)$ may be asymptotically linear in t just as the cumulants of $N(t)$ are. Smith has shown this for two special kinds of cumulative processes. $W(t)$ is a Type A cumulative process if

$$W(t) = \sum_{k=1}^{N(t)+1} Y_k. \tag{1.2.6}$$

It is a Type B process if

$$W(t) = \begin{cases} 0 & , \ t < X_1 \\ \sum_{k=1}^{N(t)} Y_k, & t \geq X_1 \ . \end{cases} \qquad (1.2.7)$$

The advantage of dealing with Type A and B processes rather than the more general process

$$W(t) = \sum_{k=1}^{N(t)} Y_k + \xi(t - S_{N(t)}, \omega_{N(t)+1})$$

is that we thereby avoid questions about the behavior of the process between regeneration points.

Theorem (Smith 1979)

Suppose $W(t)$ is either a Type A or Type B cumulative process. For integer $n \geq 1$, assume $EX_1^{n+p+1} < \infty$, $E|Y_1|^n < \infty$, and $EX_1^{r+p}|Y_1|^s < \infty$ for $r \leq n$, $s \leq n$, $r + s \leq n + 1$, and $p \geq 0$. Then the nth cumulant of $W(t)$ is

$$k_n(t) = A_n t + B_n + \frac{\lambda(t)}{(1+t)^p} \ .$$

The function $\lambda(t)$ is of bounded total variation, is $o(1)$ as $t \to \infty$, and if $p \geq 1$, $\frac{\lambda(t)}{1+t}$ belongs to the class L_1.

1.3 TRANSIENT RENEWAL PROCESSES

All of the quoted properties of renewal and cumulative processes depend on the assumption that $\lim_{x \to \infty} F(x) = 1$. When

$\lim_{x\to\infty} F(x) = \omega < 1$, they no longer hold. We allow X to take the value "∞" with probability $1 - \omega$ and when $X_n = \infty$ we say the renewal process "dies" at time S_{n-1}. The probability an infinite lifetime eventually occurs is

$$P\{X_n = \infty, \text{ some } n\} = \sum_{n=1}^{\infty} P\{X_1, \ldots, X_{n-1} \text{ all finite; } X_n = \infty\}$$

$$= (1-\omega) \sum_{n=1}^{\infty} \omega^{n-1} = 1. \qquad (1.3.1)$$

The process dies with probability one and is called a transient renewal process for that reason.

If $X_1, X_2, \ldots, X_{N(t)+1}$ are all finite, we say the process is "alive" at time t. Let

$$A(t) = \begin{cases} 1 \text{ if the process is alive at } t \\ 0 \text{ otherwise} \end{cases}$$

and $q(t) = P\{A(t) = 1\}$.

$$q(t) = \sum_{n=0}^{\infty} P\{A(t) = 1, N(t) = n\}$$

$$= \omega - F(t) + \int_0^t [\omega - F(t-\tau)]dH(\tau) = \omega - (1-\omega)H(t). \qquad (1.3.2)$$

As usual, $H(t) = \sum_{n=1}^{\infty} F^{(n)}(t)$ and the integral equation $H(t) = F(t) + \int_0^t H(t-\tau)dF(\tau)$ is still true despite the transient nature of the process. Note, however, that

$$\lim_{t\to\infty} H(t) = \sum_{n=1}^{\infty} \omega^n = \frac{\omega}{1-\omega} . \tag{1.3.3}$$

We only expect to see $\frac{\omega}{1-\omega}$ renewals before the process dies.

The fact that $q(t) \to 0$ as $t \to \infty$ explains why the usual renewal theoretic results fail to hold when F is defective. W(t) cannot be defined satisfactorily and "asymptotic normality" makes no sense in this case. Because H(t) is not asymptotically linear in t but converges to a finite limit instead, a Key Renewal type result is not possible. Rather than finding

$$\int_0^x Q(x-\tau)dH(\tau) \to \frac{1}{\mu_1} \int_0^\infty Q(\tau)d\tau ,$$

we find

$$\int_0^x Q(x-\tau)dH(\tau) \to 0 \quad \text{as } x \to \infty$$

for all functions Q satisfying the conditions of the Key Renewal Theorem. Informally, for large t, dH(t) acts like "$0\cdot dt$" and not like "$\mu_1^{-1}\cdot dt$."

1.4 EXAMPLES OF TRANSIENT RENEWAL AND CUMULATIVE PROCESSES

1. Time Until Ruin in Collective Risk Theory

Let the renewal process $\{X_i\}$ represent times between claims against an insurance company and let Y_j be the value of the jth claim. Assume that in the absence of claims, reserves increase at the constant rate $c > 0$ and that the company has initial assets u.

The risk reserve at time t is

$$R(t) = u + ct - \sum_{j=1}^{N(t)} Y_j.$$

The company is "ruined" at time t if $R(t) < 0$. Define

$$\tau^* = \inf\{t : R(t) < 0\} \tag{1.4.1}$$

Cramer (1955), von Bahr (1974), Siegmund (1975), and others have studied τ^*, the time of ruin.

Let

$$W(t) = \sum_{j=1}^{N(t)} Y_j - ct;$$

$W(t)$ is a cumulative process with increments $Y_j - cX_j$. Ruin can occur only at some regeneration point S_n when $W(t)$ exceeds all of its previous values. Let $L_1, L_2, \ldots$ be the ladder random variables constructed from the increments $Y_j - cX_j$. That is,

$$L_1 = \sum_{k=1}^{I_1} (Y_k - cX_k)$$

if I_1 is the smallest integer making L_1 positive. For $n = 2, 3, \ldots,$

$$L_n = \sum_{k=I_1+\cdots+I_{n-1}+1}^{I_1+\cdots+I_n} (Y_k - cX_k)$$

where I_n is the smallest integer making L_n positive. L_n is the sum of I_n iid random variables; the L's are iid because the I's are.

Associated with the L's is the renewal process $\{Z_i\}$,

$$Z_1 = \sum_{k=1}^{I_1} X_k$$

$$Z_n = \sum_{k=I_1+\cdots+I_{n-1}+1}^{I_1+\cdots+I_n} X_k \qquad n = 2, 3, \ldots$$

Let M(t) be the renewal count for the process $\{Z_i\}$ and let

$$W_Z(t) = \sum_{k=1}^{M(t)} L_k$$

be the increasing cumulative process built from the ladder variables.

$$P\{\tau^* \le t\} = P\{\sum_{k=1}^{M(t)} L_k > u\}. \tag{1.4.2}$$

If $E(Y_1-cX) < 0$, $\sum_{k=1}^{n} (Y_k-cX_k)$ attains a maximum with probability one and then drifts toward $-\infty$. This means $\{Z_i\}$ and $W_Z(t)$ are transient processes. In Chapter 4 we will suggest a way of coping with (1.4.2); our results and methods are similar to those of von Bahr and Siegmund.

2. Waiting Times for Long Gaps

Suppose the renewal process $\{X_i\}$ is stopped at the first appearance of an interval longer than L which is free of renewals. Let F be the proper distribution of the X's and define W to be the waiting time for such an interval. Let $V(t) = P\{W \le t\}$. $V(t) = 0$, $t \le L$. For $t > L$, the event $\{W \le t\}$ can occur in one of two ways.

Either X_1 itself exceeds L or $X_1 \leq L$ and the search for a long gap begins again from the new starting point X_1 and the event $\{W \leq t - X_1\}$ occurs. Thus

$$V(t) = 1-F(L) + \int_0^L V(t-\tau)dF(\tau), \quad t>L. \tag{1.4.3}$$

We can write

$$V(t) = v(t) + \int_0^t V(t-\tau)dG(\tau) \tag{1.4.4}$$

where G is a defective distribution defined by

$$G(t) = \begin{cases} F(t), & t \leq L \\ F(L), & t > L \end{cases}$$

and

$$v(t) = \begin{cases} 0 & ; t \leq L \\ 1-F(L), & t > L. \end{cases}$$

Hence

$$V(t) = v(t) + \int_0^t v(t-\tau)dH_G(\tau). \tag{1.4.5}$$

Feller (1971) uses the notion of waiting for a large gap to model the problem of a pedestrian trying to cross a stream of traffic. Let the renewal process $\{X_i\}$ be the gaps between successive cars. In order to cross the street with safety, the pedestrian must wait

for a gap of more than L seconds, say. The distribution of his waiting time is given by (1.4.5).

$V(t)$ may also be interpreted as the probability that the maximum lifetime or partial lifetime observed by time t exceeds L. Lamperti (1961) has studied the problem from this point of view.

3. Lost Telephone Calls

Suppose that calls arriving at a telephone trunkline form a Poisson process with intensity λ. A call is placed at time 0. The lengths of conversations are independent random variables with common distribution F. Calls arriving during a busy period are lost; we are interested in the waiting time W for the first lost call. We may consider the renewal process $\{X_i\}$ where X_i is the time between the i-1st and ith calls so long as no calls arrive during the busy period caused by the i-1st call; otherwise $X_i = \infty$ and the process stops.

The busy periods associated with the process have distribution

$$G(x) = \int_0^x e^{-\lambda\tau} dF(\tau). \tag{1.4.6}$$

The event $\{X_1 \le x\}$ occurs when the call begun at time 0 lasts for some time $\tau \le x$ and a new call is received in the remaining time $x - \tau$. Hence

$$J(x) = P\{X_1 \le x\} = \int_0^x [1-e^{-\lambda(x-\tau)}]e^{-\lambda\tau} dF(\tau) \tag{1.4.7}$$

$$\lim_{x\to\infty} J(x) = \int_0^\infty e^{-\lambda\tau} dF(\tau) = F^*(\lambda) = \omega < 1. \tag{1.4.8}$$

We can study W, the waiting time for the first lost call, by studying the transient process $\{X_i\}$. Let q(t) be the probability the X process is alive at time t.

$$P\{W>t\} = q(t) = \omega - (1-\omega)H_J(t) \tag{1.4.9}$$

by (1.3.2). Of course, we know that $P\{W>t\} \to 0$ as $t \to \infty$, but it would be interesting to know the rate of this convergence. Results obtained in Chapters 3 and 4 address this question. This example is from problem 17, Chapter 6.13 of Feller (1971).

4. Generalized Type II Geiger Counters

Particles arriving at a generalized type II Geiger counter constitute a Poisson process with intensity λ. The n<u>th</u> particle locks the counter for a time T_n and annuls the after-effects of all preceding particles. Suppose the T's have common distribution B and let Y_i be the length of the i<u>th</u> locked period. Define $Z(t) = P\{Y_1>t\}$. The event $\{Y_1>t\}$ can result either because $T_1 > t$ and no particles arrive in (0,t] or because a particle arrives at some time $\tau < T_1$, $\tau \le t$ and the locked period begun at τ exceeds $t - \tau$. Thus

$$Z(t) = e^{-\lambda t}[1-B(t)] + \int_0^t \lambda e^{-\lambda\tau}[1-B(\tau)]Z(t-\tau)d\tau. \tag{1.4.10}$$

We may regard

$$F(t) = \int_0^t [1-B(\tau)]\lambda e^{-\lambda\tau}d\tau \tag{1.4.11}$$

as a defective distribution and write

$$Z(t) = e^{-\lambda t}[1-B(t)] + \int_0^t Z(t-\tau)dF(\tau) \tag{1.4.12}$$

which implies

$$Z(t) = e^{-\lambda t}[1-B(t)] + \int_0^t e^{-\lambda(t-\tau)}[1-B(t-\tau)]dH_F(\tau). \tag{1.4.13}$$

To investigate the distribution of the locked periods, we must deal with the transient renewal function $H_F(t)$. Also, if we are interested in the renewal process $\{X_i\}$ where X_i represents the time between the beginning of the i-1st and ith blocked periods, we see that

$$P\{X_1 \le x\} = \int_0^x \lambda e^{-\lambda\tau}[1-Z(x-\tau)]d\tau \tag{1.4.14}$$

because X_1 is the sum of the length of a locked period and the waiting time for the arrival of the first particle after the counter has become unlocked. Study of $\{X_i\}$ requires coping with F and H_F. This example is from problem 15, Chapter 11.10 of Feller (1971).

5. Age Dependent Branching Processes

A particle born at time 0 lives some random time and then splits into k new particles with probability q_k, k = 0, 1, ... Its life-time has distribution G; assume G(0+) = 0 and G(∞) = 1. The new particles develop independently of one another and of their time of

birth; they have the same lifetime distribution and splitting probabilities as the first one.

Let $Z(t)$ be the number of particles at time t and $p_r(t) = P\{Z(t) = r\}$. If $Z(t) = 0$, then $Z(t+s) = 0$ for all $s \geq 0$; the branching process becomes extinct. Define the generating functions

$$h(s) = \sum_{n=0}^{\infty} q_n s^n \quad \text{and} \quad F(s,t) = \sum_{r=0}^{\infty} p_r(t) s^r. \tag{1.4.15}$$

Bellman and Harris (1948) initiated the study of age dependent branching processes and derived the equation

$$F(s,t) = \int_0^t h[F(s,t-y)]dG(y) + s[1-G(t)]. \tag{1.4.16}$$

Levinson (1960) proved that (1.4.16) has a unique solution $F(s,t)$ which is a generating function for each t and is the unique bounded solution.

To investigate $A(t)$, the expected number of particles at time t, suppose $\alpha = h'(1) < \infty$.

$$A(t) = \left.\frac{\partial F(s,t)}{\partial s}\right|_{s=1} = \alpha\int_0^t A(t-y)dG(y) + 1 - G(t). \tag{1.4.17}$$

Bondarenko (1960) has shown that if $\alpha \leq 1$, $p_0(t) \to 1$ as $t \to \infty$. That is, the process becomes extinct with probability one. In particular, consider the case $\alpha < 1$. We may regard $\alpha G(y)$ as a defective distribution with corresponding renewal function $H_\alpha(y)$. Hence

$$A(t) = 1-G(t) + \int_0^t [1-G(t-\tau)]dH_\alpha(\tau). \tag{1.4.18}$$

Depending upon our assumptions about the tail of G, we can find different asymptotic estimates of A(t) by using the results of Chapters 3 and 4. These estimates agree with those of Vinogradov (1964) and Chistyakov (1964).

To study $p_0(t)$ and $u(t) = 1-p_0(t)$, we need only notice that $p_0(t) = F(0,t)$. Thus equation (1.4.16) yields

$$p_0(t) = \int_0^t h[p_0(t-y)]dG(y). \tag{1.4.19}$$

Vinogradov and Chistyakov have derived asymptotic expressions for u(t) from (1.4.19) by expanding h in a power series around 1 and applying methods like those we will discuss in Chapters 3 and 4.

CHAPTER 2

PRELIMINARY RESULTS ON TRANSIENT PROCESSES

2.1 INTRODUCTION

Our goal is to develop a theory for transient renewal and cumulative processes which parallels that for standard renewal and cumulative processes. To ensure that it makes sense to talk about the behavior of a transient process at time t, we will condition expectations and probabilities of interest upon $\{A(t) = 1\}$. We will study questions of the following type:

1) Does $P\{A(t+s) = 1|A(t) = 1\}$ have a limit as $t \to \infty$?

2) Do $P\{\zeta_t \le x|A(t) = 1\}$ and $P\{\eta_t \le x|A(t) = 1\}$ have limits which are proper distributions in x?

3) Is $E[W(t)^k|A(t) = 1]$ asymptotically a k<u>th</u> degree polynomial in t?

4) Is W(t) conditionally normal?

A brief examination of several of these questions will indicate the types of expressions we must cope with if we are to answer these queries satisfactorily.

Notice that

$$P\{A(t+s)|A(t)=1\} = \frac{q(t+s)}{q(t)} = \frac{H(\infty) - H(t+s)}{H(\infty) - H(t)} .$$

It is not difficult to derive the conditional distribution of the backward delay. We have that

$$P\{\eta_t \le x,\ A(t)=1\} = \sum_{n=0}^{\infty} P\{\eta_t \le x,\ A(t)=1,\ N(t)=n\}$$

$$= U(x-t)[\omega-F(t)] + \sum_{n=1}^{\infty} \int_{t-x}^{t} [\omega-F(t-\tau)]dF^{(n)}(\tau)$$

$$= U(x-t)[\omega-F(t)] + \int_{t-x}^{t} [\omega-F(t-\tau)]dH(\tau).$$

Therefore

$$P\{\eta_t \le x | A(t)=1\} = \frac{U(x-t)[\omega-F(t)]+\int_{t-x}^{t} [\omega-F(t-\tau)]dH(\tau)}{(1-\omega)[H(\infty)-H(t)]}. \qquad (2.1.1)$$

Taking $W(t) = N(t)$ and $k = 1$ in question 3 leads us to investigate $E[N(t)|A(t) = 1]$. But

$$EN(t)A(t) = E\sum_{n=1}^{\infty} \chi(S_n \le t,\ A(t)=1)$$

$$= \sum_{n=1}^{\infty} \int_0^t q(t-\tau)dF^{(n)}(\tau) = \int_0^t q(t-\tau)dH(\tau).$$

Hence

$$E[N(t)|A(t)=1] = \int_0^t \frac{[H(\infty)-H(t-\tau)]dH(\tau)}{H(\infty)-H(t)}. \qquad (2.1.2)$$

These examples demonstrate that to answer the questions posed we must study $F(\infty)-F(t)$, $H(\infty)-H(t)$, and their relationship.

2.2 THE TAIL OF H

We begin with a helpful lemma.

Lemma 2.1

$q(t+s) \geq q(t)q(s)$ for all $t,s > 0$.

Proof:

Let $G_t(x) = P\{\zeta_t \leq x\}$

$$q(t+s) = P\{A(t+s)=1\} = P\{A(t+s)=1, \zeta_t \leq s\} + P\{A(t+s)=1, \zeta_t > s\}$$

$$= \int_0^s q(s-u)dG_t(u) + P\{A(t+s) = 1, \zeta_t > s\}$$

$$\geq q(s)[G_t(s) + P\{A(t) = 1, \zeta_t > s\}] = q(s)q(t). \qquad \square$$

It is well known (see, for example, Chapter 1 of Galambos and Kotz (1978)) that if

$$\psi(s) = \lim_{t\to\infty} \frac{q(t+s)}{q(t)}$$

exists, then either

(i) $\psi(s) = e^{-\sigma s}$ for some $\sigma > 0$ or

(ii) $\psi(s)$ is either 0 or 1 for all $s > 0$.

In fact, Lemma 2.1 shows that in case ii), $\psi(s) \equiv 1$ because $\psi(s) \geq q(s) > 0$.

If $\lim_{t\to\infty} \frac{f(t+s)}{f(t)} = 1$ for all fixed s, f is called a function of moderate growth. All such functions have the property that

$$f(t) \sim k \exp \{\int_0^t \alpha(u)du\} \tag{2.2.1}$$

where $\alpha(u) \to 0$ as $u \to \infty$. See Smith (1972).

Hence if $\psi(s)$ exists, either $H(\infty)-H(t)$ or $e^{\sigma t}[H(\infty)-H(t)]$ is a function of moderate growth. When case i) obtains, if we fix $\varepsilon > 0$, there is a $T(\varepsilon)$ such that

$$e^{\sigma t}[H(\infty)-H(t)] < e^{\varepsilon t} \quad \text{for all} \quad t \geq T.$$

Thus if we choose any $0 < c < \sigma$ and fix $\varepsilon < \sigma - c$,

$$\int_0^\infty e^{ct}[H(\infty)-H(t)]dt$$

$$< \int_0^T e^{ct}[H(\infty)-H(t)]dt + \int_T^\infty e^{-(\sigma-c-\varepsilon)t}dt < \infty . \tag{2.2.2}$$

This ensures that both H and F have moments of all orders.

We shall see in Chapter 4 that σ is an important constant and $\int_0^\infty e^{\sigma x}dF(x) \leq 1$. If the equality holds, we can salvage many standard renewal theoretic results by conditioning arguments. However, if the inequality is strict, much less can be proved. But regardless of whether $F^*(-\sigma) = 1$ or $F^*(-\sigma) < 1$, we have the following lemma.

<u>Lemma 2.2</u>

Suppose ϕ is an integrable function on $[0,x]$. If $\frac{q(t+s)}{q(t)} \to e^{-\sigma s}$,

then

$$\lim_{t\to\infty} \int_{t-x}^{t} \frac{\phi(t-\tau)}{q(t)}\, dH(\tau) = \frac{\sigma}{1-\omega} \int_0^x e^{\sigma y}\phi(y)dy \qquad (2.2.3)$$

Proof:

Establish a partition of $[t-x,t]: t-x = y_0 < y_1 < \cdots < y_N = t$.

Let $\min\limits_{y_j \le \tau \le y_{j+1}} \phi(t-\tau) = \underline{\phi}_j$

$$\max_{y_j \le \tau \le y_{j+1}} \phi(t-\tau) = \overline{\phi}_j$$

$$\sum_{j=0}^{N-1} \underline{\phi}_j \;\frac{H(y_{j+1})-H(y_j)}{H(\infty) - H(y_j)} \quad \frac{H(\infty) - H(y_j)}{(1-\omega)(H(\infty)-H(t))} \le \int_{t-x}^{t} \frac{\phi(t-\tau)}{q(t)}\, dH(\tau)$$

$$\le \sum_{j=0}^{N-1} \overline{\phi}_j \quad \frac{H(y_{j+1})-H(y_j)}{H(\infty)- H(y_j)} \quad \frac{H(\infty) - H(y_j)}{(1-\omega)(H(\infty)-H(t))} \quad .$$

As

$$t\to\infty,\ \frac{H(\infty)-H(y_j)}{H(\infty)-H(t)} \to e^{\sigma(t-y_j)}$$

and

$$\frac{H(y_{j+1})-H(y_j)}{H(\infty)-H(y_j)} \to 1 - e^{-\sigma(y_{j+1}-y_j)};$$

this convergence is uniform since $t - x \leq y_j \leq t$. Thus for any $\varepsilon < 0$, we can choose t to be large enough to make

$$\frac{1}{1-\omega}\sum_{j=0}^{N-1} \underline{\phi}_j\{[1-e^{-\sigma(y_{j+1}-y_j)}]e^{\sigma(t-y_j)} - \varepsilon\} \leq \int_{t-x}^{t} \frac{\phi(t-\tau)}{q(t)}\, dH(\tau)$$

$$\leq \frac{1}{1-\omega}\sum_{j=0}^{N-1} \overline{\phi}_j\{[1-e^{-\sigma(y_{j+1}-y_j)}]e^{\sigma(t-y_j)} + \varepsilon\}.$$

Letting the mesh of the partition approach zero and noting that $\phi(y)e^{\sigma y}$ is necessarily integrable on $[0,x]$, we conclude

$$\lim_{t\to\infty} \int_{t-x}^{t} \phi(t-\tau)dH(\tau) = \frac{\sigma}{1-\omega}\int_{t-x}^{t} \phi(t-\tau)e^{\sigma(t-\tau)}d\tau$$

$$= \frac{\sigma}{1-\omega}\int_0^x \phi(y)e^{\sigma y}dy. \qquad \square$$

<u>Corollary 2.1</u>

If $\frac{q(t+s)}{q(t)} \to e^{-\sigma s}$, then

$$\lim_{t\to\infty} P\{\eta_t \leq x | A(t)=1\} = \frac{\sigma}{1-\omega}\int_0^x [\omega-F(y)]e^{\sigma y}dy \equiv J(x). \tag{2.2.4}$$

Proof:

From (2.1.1), we have

$$P\{\eta_t \leq x | A(t)=1\} = \frac{U(x-t)\,[\omega-F(t)] + \int_{t-x}^{t} [\omega-F(t-\tau)]dH(\tau)}{q(t)}.$$

The result follows directly.

Corollary 2.2

If $\frac{q(t+s)}{q(t)} \to e^{-\sigma s}$, then

$$\lim_{t\to\infty} P\{\zeta_t \le x | A(t)=1\} = \frac{\sigma e^{-\sigma x}}{1-\omega} \int_0^x e^{\sigma y}[1-F(y)]dy \equiv L(x). \tag{2.2.5}$$

Proof:

$$P\{\zeta_t \le x, A(t)=1\} = \sum_{n=0}^{\infty} P\{\zeta_t \le x, A(t)=1, N(t)=n\}$$

$$= F(t+x) - F(t) + \sum_{n=1}^{\infty} \int_0^t [F(t+x-\tau)-F(t-\tau)]dF^{(n)}(\tau)$$

$$= F(t+x)-F(x) + \int_0^{t+x} F(t+x-\tau)dH(\tau) - \int_t^{t+x} F(t+x-\tau)dH(\tau)$$

$$- \int_0^t F(t-\tau)dH(\tau)$$

$$= F(t+x)-F(t)+H(t+x)-F(t+x) - \int_t^{t+x} F(t+x-\tau)dH(\tau)-H(t)+F(t)$$

$$= \int_t^{t+x} [1-F(t+x-\tau)]dH(\tau).$$

Therefore

$$P\{\zeta_t \le x | A(t)=1\} = \int_x^{t+x} \frac{[1-F(t+x-\tau)]dH(\tau)}{q(t)} \tag{2.2.6}$$

$$\lim_{t\to\infty} P\{\zeta_t \le x | A(t)=1\} = \lim_{t\to\infty} \int_t^{t+x} \frac{[1-F(t+x-\tau)]dH(\tau)}{q(t+x)} \cdot \frac{q(t+x)}{q(t)}$$

$$= \frac{\sigma e^{-\sigma x}}{1-\omega} \int_0^x e^{\sigma y}[1-F(y)]dy. \qquad \square$$

We can rewrite the limiting distribution as

$$L(x) = (1-\omega)^{-1}\{1-\omega-e^{-\sigma x}+ e^{-\sigma x} \int_0^x e^{\sigma y}dF(y)+\omega-F(x)\}. \tag{2.2.7}$$

It is worth noting that J and L are distinct distributions. In proper renewal theory, the forward and backward delays share the same limiting distribution. In the transient setting, however, η_t is asymptotically greater in distribution than ζ_t.

<u>Lemma 2.3</u>

$L(x) \geq J(x)$ for all $x \geq 0$.

Proof:

Using the fact that $\int_0^\infty e^{\sigma x}dF(x) \leq 1$, we have

$$1-e^{-\sigma x} \geq (1-e^{-\sigma x}) \{\int_0^x e^{\sigma z}dF(z) + \int_x^\infty e^{\sigma z}dF(z)\}$$

$$\geq (1-e^{-\sigma x}) \int_0^x e^{\sigma z}dF(z) + \int_x^\infty (e^{\sigma x}-1)dF(z).$$

Thus

$$1-e^{-\sigma x} + e^{-\sigma x} \int_0^x e^{\sigma z}dF(z) + \omega-F(x) \geq \int_0^x e^{\sigma z}dF(z) + \int_x^\infty e^{\sigma x}dF(z)$$

$$\Leftrightarrow 1-e^{-\sigma x} + e^{-\sigma x}\int_0^x e^{\sigma z}dF(z) + \omega - F(x) \geq$$

$$\sigma\int_0^x\int_0^z e^{\sigma y}dy\ dF(z) + \sigma\int_x^\infty\int_0^x e^{\sigma y}dy\ dF(z) + \omega$$

$$\Leftrightarrow 1-\omega-e^{-\sigma x} + e^{-\sigma x}\int_0^x e^{\sigma z}dF(z) + \omega - F(x) \geq$$

$$\sigma\int_0^x\int_y^\infty e^{\sigma y}dF(z)dy = \sigma\int_0^x e^{\sigma y}[\omega - F(y)]dy.$$

From (2.2.4) and (2.2.7), we see $L(x) \geq J(x)$. $\square$

Much of proper renewal theory draws heavily upon the fact that $H(t) = EN(t)$ is asymtotically linear in t. In the transient situation, if $H(\infty) - H(t)$ is a function of moderate growth, conditioning on $\{A(t) = 1\}$ will not help us develop an analogous theory. This is because $E[N(t)|A(t) = 1]$ is not asymptotically linear.

<u>Lemma 2.4</u>

If $H(\infty) - H(t)$ is a function of moderate growth, then
$$\frac{E[N(t)|A(t)=1]}{t} \to 0.$$

Proof:

From (2.1.2) we have

$$E[N(t)|A(t)=1] = \int_0^t \frac{q(t-\tau)}{q(t)}\,dH(\tau).$$

Fix $\varepsilon > 0$. There exists an $N(\varepsilon)$ such that $\frac{H(x+1)-H(x)}{H(\infty)-H(x)} < \varepsilon$ for all $x \geq N$.

$$E[N(t)|A(t)=1] = \int_0^N \frac{q(t-\tau)}{q(t)}\, dH(\tau) + \int_N^t \frac{q(t-\tau)}{q(t)}\, dH(\tau).$$

By Lemma 2.1, $\frac{q(t-\tau)}{q(t)} \leq \frac{1}{q(\tau)}$. Thus

$$E[N(t)|A(t)=1] \leq \frac{q(t-N)}{q(t)} H(N) + \int_N^t \frac{dH(\tau)}{q(\tau)}.$$

But

$$\int_N^t \frac{dH(\tau)}{q(\tau)} \leq \sum_{j=N}^{[t]} \int_j^{j+1} \frac{dH(\tau)}{q(\tau)} \leq \frac{1}{1-\omega} \sum_{j=N}^{[t]} \frac{H(j+1)-H(j)}{H(\infty)-H(j)} \leq \frac{t\varepsilon}{1-\omega}.$$

Hence

$$\frac{E[N(t)|A(t)=1]}{t} \to 0$$

since ε is arbitrary. □

2.3 THE RELATIONSHIP BETWEEN $H(\infty)-H(t)$ AND $F(\infty)-F(t)$

We now turn to the relationship between the moments of F and H and between the rates at which the tails of F and H approach 0. Let $\mu_k = \int_0^\infty x^k dF(x) \leq \infty$ and $\nu_k = \int_0^\infty x^k dH(x) \leq \infty$.

The integral equation of renewal theory yields a simple relation between the Laplace-Stieltjes transforms of F and H.

$$H^*(s) = \frac{F^*(s)}{1-F^*(s)}. \tag{2.3.1}$$

Lemma 2.5

H and F have the same number of finite moments.

Proof:

$$\int_0^\infty x^k dH(x) < \infty \Rightarrow \int_0^\infty x^k dF(x) < \infty .$$

Suppose $\int_0^\infty x^k dF(x) < \infty$. Then $\frac{d^k}{ds^k} F^*(s)$ exists and is finite for $s \geq 0$. Hence

$$\frac{d^k}{ds^k} H^*(s) = \sum_{j=0}^{k} \binom{k}{j} \frac{d^j}{ds^j} F^*(s) \frac{d^{k-j}}{ds^{k-j}} \left[\frac{1}{1-F^*(s)}\right] \tag{2.3.1}$$

exists and is finite for $s \geq 0$. (Note that $F^*(s) \leq \omega < 1$ for $s \geq 0$.) Therefore

$$\left.\frac{d^k}{ds^k} H^*(s)\right|_{s=0} = (-1)^k \nu_k \text{ is finite.} \qquad \square$$

From expression (2.3.1), we see that ν_k is a linear combination of terms like $c\mu_{k_1} \cdots \mu_{k_m}$ where $\sum_{n=1}^{m} k_n = k$. In fact,

$$\nu_k = \frac{\mu_k}{(1-\omega)^2} + \text{terms involving products of lower order moments of F.}$$

Because of this form, one might expect that if $\mu_{r+1} = \infty$, then

$$\lim_{x\to\infty} \frac{\int_0^x y^r [H(\infty)-H(y)]dy}{\int_0^x y^r [F(\infty)-F(y)]dy} = \frac{1}{(1-\omega)^2} .$$

We will show that this is indeed the case.

We begin with a variation on Feller's (1963) ratio limit theorem.

Theorem 2.1

Let $a(x)$ and $b(x)$ be non-negative and non-increasing functions of $x \in [0,\infty)$ and define $\phi(x) = x^k a(x)$ and $\psi(x) = x^k b(x)$, $k > 0$. Suppose $\phi^0(s)$ and $\psi^0(s)$ exist for all $s > 0$. Then if we write

$$\Phi(x) = \int_0^x \phi(y)dy \text{ and } \Psi(x) = \int_0^x \psi(y)dy,$$

$$\frac{\Phi(x)}{\Psi(x)} \to \alpha \text{ as } x \to \infty \text{ if and only if}$$

$$\frac{\phi^0(s)}{\psi^0(s)} \to \alpha \text{ as } s \to 0 \qquad (\alpha < \infty).$$

Proof I:

Assume $\dfrac{\phi^0(s)}{\psi^0(s)} \to \alpha$ as $s \to 0$. Define

$$\phi_t^c(x) = \frac{e^{-cx} t\phi(tx)}{\phi^0(c/t)}\ ; \quad \Phi_t^c(x) = \int_0^x \phi_t^c(y)dy$$

and

$$\psi_t^c(x) = \frac{e^{-cx} t\psi(tx)}{\psi^0(c/t)}\ ; \quad \Psi_t^c(x) = \int_0^x \psi_t^c(y)dy.$$

Let

$$\overline{\lim_{t\to\infty}} \frac{\Phi(t)}{\Psi(t)} = \lambda \le \infty.$$

Then there is a sequence

$$\{t_j\} \uparrow \infty \quad \text{such that} \quad \frac{\Phi(t_j)}{\Psi(t_j)} \to \lambda \; .$$

By Helly's Selection Theorem, $\exists$ a subsequence $\{t_{j_n}\}$ such that

$$\Phi^c_{t_{j_n}}(x) \xrightarrow{w} F_1(x) \text{ and } \Psi^c_{t_{j_n}}(x) \xrightarrow{w} F_2(x)$$

where F_1 and F_2 are nondecreasing right continuous functions bounded between 0 and 1. The continuity theorem for Laplace transforms implies

$$\Phi^{c^0}_{t_{j_n}}(s) \to F^0_1(s) \quad \text{and} \quad \Psi^{c^0}_{t_{j_n}}(s) \to F^0_2(s).$$

Upon calculating transforms, we see this means

$$\frac{\phi^0(\frac{c+s}{t_{j_n}})}{s\phi^0(\frac{c}{t_{j_n}})} \to F^0_1(s) \text{ and } \frac{\psi^0(\frac{c+s}{t_{j_n}})}{s\psi^0(\frac{c}{t_{j_n}})} \to F^0_2(s). \qquad (2.3.2)$$

We want to be sure that $F_1{}^0(s)$ and $F_2{}^0(s)$ are strictly greater than zero for all $s \geq 0$.

Note that

$$\Phi^c_{t_{j_n}}(x) = \frac{\int_0^x e^{-cy} t_{j_n}^{k+1} y^k a(t_{j_n} y)dy}{\int_0^\infty e^{-cy} t_{j_n}^{k+1} y^k a(t_{j_n} y)dy}$$

$$= 1 - \frac{\int_x^\infty e^{-cy} t_{j_n}^{k+1} y^k a(t_{j_n} y)\,dy}{\int_0^\infty e^{-cy} t_{j_n}^{k+1} y^k a(t_{j_n} y)\,dy}$$

$$\geq 1 - \frac{a(t_{j_n} x)\int_x^\infty e^{-cy} y^k dy}{a(t_{j_n} x)\int_0^x e^{-cy} y^k dy}$$

$$= 1 - \frac{\left[\frac{\Gamma(k+1)}{c^{k+1}} - \int_0^x e^{-cy} y^k dy\right]}{\int_0^x e^{-cy} y^k dy}$$

$$= 2 - \frac{\Gamma(k+1)}{c^{k+1}\int_0^x e^{-cy} y^k dy} .$$

This is strictly positive for sufficiently large x.

Therefore $F_1(x) > 0$ for all large x and hence $F_1^0(s) > 0$ for all $s \geq 0$. By exactly the same argument, $F_2^0(s) > 0$ for all $s \geq 0$. From (2.3.2) we see

$$\frac{\phi^0\left(\frac{c+s}{t_{j_n}}\right)}{\psi^0\left(\frac{c+s}{t_{j_n}}\right)} \; \frac{\psi^0\left(\frac{c}{t_{j_n}}\right)}{\phi^0\left(\frac{c}{t_{j_n}}\right)} \to \frac{F_1{}^0(s)}{F_2{}^0(s)} .$$

But by assumption, the product of these factors converges to 1 as $t_{j_n} \to \infty$.

Hence $F_1^0(s) = F_2^0(s) \equiv F^0(s)$.

Let $z > 1$ be a continuity point of $F(x)$. Then

$$\frac{\Phi^c_{t_{j_n}}(z)}{\Psi^c_{t_{j_n}}(z)} \to 1 .$$

And

$$\frac{\Phi^c_{t_{j_n}}(z)}{\Psi^c_{t_{j_n}}(z)} \cdot \frac{\phi^0(\frac{c}{t_{j_n}})}{\psi^0(\frac{c}{t_{j_n}})} \to \alpha ;$$

i.e.,

$$\frac{\int_0^z e^{-cy} t_{j_n} \phi(t_{j_n} y)dy}{\int_0^z e^{-cy} t_{j_n} \psi(t_{j_n} y)dy} \to \alpha .$$

Thus

$$\overline{\lim_{n\to\infty}} \frac{\int_0^1 e^{-cy} t_{j_n} \phi(t_{j_n} y)dy}{\int_0^z e^{-cy} t_{j_n} \psi(t_{j_n} y)dy} \le \alpha$$

$$\overline{\lim_{n\to\infty}} \frac{e^{-c}\int_0^1 t_{j_n} \phi(t_{j_n} y)dy}{\int_0^z t_{j_n} \psi(t_{j_n} y)dy} \le \alpha$$

$$\overline{\lim_{n\to\infty}} \frac{e^{-c}\int_0^{t_{j_n}} \phi(u)du}{z\int_0^{t_{j_n}} \psi(zu)du} \le \alpha$$

$$\overline{\lim_{n\to\infty}} \frac{e^{-c}\int_0^{t_{j_n}} \phi(u)du}{z^{k+1}\int_0^{t_{j_n}} u^k b(zu)du} \le \alpha$$

$$\overline{\lim_{n\to\infty}} \frac{\int_0^{t_{j_n}} \phi(u)du}{\int_0^{t_{j_n}} u^k b(u)du} \le e^c z^{k+1}\alpha$$

$$\Rightarrow \lambda \le e^c z^{k+1}\alpha.$$

But c is arbitrary and z is arbitrarily close to 1. Therefore $\lambda \le \alpha$. By reversing the roles of Φ and Ψ we will find

$$\overline{\lim_{t\to\infty}} \frac{\Psi(t)}{\Phi(t)} \le \alpha^{-1} \Rightarrow \underline{\lim_{t\to\infty}} \frac{\Phi(t)}{\Psi(t)} \ge \alpha.$$

Hence

$$\lim_{t\to\infty} \frac{\Phi(t)}{\Psi(t)} = \alpha.$$

Proof II:

We now assume that $\lim_{x\to\infty} \frac{\Phi(x)}{\Psi(x)} = \alpha$.

Let

$$\overline{\lim_{s\downarrow 0}} \frac{\phi^0(s)}{\psi^0(s)} = \lambda \le \infty .$$

Let

$$\{t_j\} \uparrow \infty \text{ be such that } \frac{\phi^0(\frac{c}{t_j})}{\psi^0(\frac{c}{t_j})} \to \lambda.$$

We can select a subsequence $\{t_{j_n}\}$ such that

$$\frac{\Phi(xt_{j_n})}{\Phi(t_{j_n})} \overset{w}{\to} F_1(x) \quad \text{and} \quad \frac{\Psi(xt_{j_n})}{\Psi(t_{j_n})} \overset{w}{\to} F_2(x) \tag{2.3.3}$$

for $0 \le x \le 1$

because $\frac{\Phi(xt)}{\Phi(t)}$ and $\frac{\Psi(xt)}{\Psi(t)}$ are d.f's on [0,1]. Let x be a continuity point of both F_1 and F_2.

$$\frac{F_1(x)}{F_2(x)} = \lim_{n\to\infty} \frac{\Phi(xt_{j_n})}{\Psi(xt_{j_n})} \cdot \frac{\Psi(t_{j_n})}{\Phi(t_{j_n})} = 1.$$

Thus $F_1(x) = F_2(x) \equiv F(x)$.

We want to be certain that $F^*(s) > 0$. Note that

$$\Phi(\tfrac{x}{2}) = \int_0^{x/2} y^k a(y)\,dy = \int_0^x (\tfrac{1}{2})^{k+1} v^k a(\tfrac{v}{2})\,dv \geq$$

$$(\tfrac{1}{2})^{k+1} \int_0^x v^k a(v)\,dv = (\tfrac{1}{2})^{k+1} \Phi(x)$$

$$\therefore \quad F(\tfrac{1}{2}) = \lim_{n\to\infty} \frac{\Phi(\frac{t_{j_n}}{2})}{\Phi(t_{j_n})} \geq (\tfrac{1}{2})^{k+1}. \quad \text{Hence } F^*(s) > 0.$$

By the continuity theorem,

$$\int_0^1 \frac{e^{-cx} d\Phi(xt_{j_n})}{\Phi(t_{j_n})}$$

and $\rightarrow F^*(c)$

$$\int_0^1 \frac{e^{-cx} d\Psi(xt_{j_n})}{\Psi(t_{j_n})}$$

and their ratio converges to 1. Thus we see

$$\frac{\int_0^1 e^{-cx} t_{j_n}^{k+1} x^k a(xt_{j_n})\,dx}{\int_0^1 e^{-cx} t_{j_n}^{k+1} x^k b(xt_{j_n})\,dx} \sim \frac{\Phi(t_{j_n})}{\Psi(t_{j_n})} \rightarrow \alpha \,. \tag{2.3.4}$$

Consider

$$\int_0^\infty e^{-cx}\; x^k a(xt_{j_n})dx = \int_0^2 e^{-cx}x^k a(xt_{j_n})dx + \sum_{m=1}^{\infty} \int_{2^m}^{2^{m+1}} e^{-cx}x^k a(xt_{j_n})dx.$$

Let

$$u = \frac{x}{2^m}\,.$$

Then

$$\int_{2^m}^{2^{m+1}} e^{-cx}x^k a(xt_{jn})dx = \int_1^2 e^{-2^m cu}(2^{k+1})^m u^k\; a(2^m ut_{j_n})du$$

$$\leq\; (2^{k+1})^m \int_1^2 e^{-[2^m-1]cu} e^{-cu}\; u^k\; a(ut_{j_n})du$$

$$\leq\; (2^{k+1})^m\; e^{-[2^m-1]c} \int_1^2 e^{-cu}\; u^k\; a(ut_{j_n})du.$$

But

$$\int_1^2 e^{-cu}\; u^k\; a(ut_{j_n})du = \int_{1/2}^1 e^{-2cu} 2^{k+1} u^k\; a(2ut_{j_n})du$$

$$\leq\; 2^{k+1}\; e^{-c/2} \int_{1/2}^1 e^{-cu} u^k a(ut_{j_n})du$$

$$\leq\; 2^{k+1}\; e^{-c/2} \int_0^1 e^{-cu}\; u^k a(ut_{j_n})du.$$

Thus

$$\int_0^\infty e^{-cx} x^k a(xt_{j_n})dx \le$$

$$\int_0^1 e^{-cx} x^k a(xt_{j_n})dx\{1 + 2^{k+1}e^{-c/2}[1 + \sum_{m=1}^\infty (2^{k+1})^m e^{-[2^m-1]c}]\}$$

$$\frac{\phi^0(\frac{c}{t_{j_n}})}{\psi^0(\frac{c}{t_{j_n}})} = \frac{\int_0^\infty e^{-cx} t_{j_n}^{k+1} x^k a(xt_{j_n})dx}{\int_0^\infty e^{-cx} t_{j_n}^{k+1} x^k b(xt_{j_n})dx}$$

$$\le \{1 + 2^{k+1}e^{-c/2}[1 + \sum_{m=1}^\infty (2^{k+1})^m e^{-[2^m-1]c}]\} \frac{\int_0^1 e^{-cx}x^k a(xt_{j_n})dx}{\int_0^1 e^{-cx}x^k b(xt_{j_n})dx}.$$

From (2.3.4), we conclude

$$\lambda \le \{1 + 2^{k+1}e^{-c/2}[1 + \sum_{m=1}^\infty (2^{k+1})^m e^{-[2^m-1]c}]\}\alpha.$$

By letting $c \to \infty$, we see $\lambda \le \alpha$.

Reversing the roles of ϕ and ψ, we will discover

$\varliminf_{s\to 0} \frac{\phi^0(s)}{\psi^0(s)} \ge \alpha$ and the conclusion follows. □

In order to apply Theorem 2.1, we derive a relation between the tails of H and F.

$$H(t) = F(t) + \int_0^t F(t-\tau)dH(\tau)$$

$$\Rightarrow \quad H(\infty)-H(t) = F(\infty)-F(t) + \frac{\omega^2}{1-\omega} - \int_0^t F(t-\tau)dH(\tau)$$

$$= F(\infty)-F(t) + \omega[H(\infty)-H(t)+H(t)] - \int_0^t F(t-\tau)dH(\tau)$$

Thus

$$(1-\omega)[H(\infty)-H(t)] = F(\infty)-F(t) + \int_0^t [\omega-F(t-\tau)]dH(\tau)$$

and hence

$$(1-\omega) \int_0^\infty e^{-st}[(H(\infty)-H(t)]dt$$

$$= \int_0^\infty e^{-st}[F(\infty)-F(t)]dt + \int_0^\infty \int_0^t e^{-st}[F(\infty)-F(t-\tau)]dH(\tau)dt$$

$$= \int_0^\infty e^{-st}[F(\infty)-F(t)]dt + \int_0^\infty e^{-s\tau} \int_\tau^\infty e^{-s(t-\tau)}[F(\infty)-F(t-\tau]dtdH(\tau).$$

Therefore

$$(1-\omega) \int_0^\infty e^{-st} [H(\infty)-H(t)]dt$$

$$=\{\int_0^\infty e^{-st}[F(\infty)-F(t)]dt\} \{1 + \int_0^\infty e^{-st}dH(\tau)\} \tag{2.3.5}$$

$$\Rightarrow \quad \lim_{s\to 0} \frac{\int_0^\infty e^{-st}[H(\infty)-H(t)]dt}{\int_0^\infty e^{-st}[F(\infty)-F(t)]dt} = \frac{1}{(1-\omega)^2} \tag{2.3.6}$$

This is true regardless of whether F and H have finite first moments.

Suppose $\mu_n < \infty$ but $\mu_{n+1} = \infty$ $(n=0, 1, \ldots)$. Then

$$(1-\omega) \int_0^\infty t^n e^{-st}[H(\infty)-H(t)]dt$$

$$= (1-\omega)\left(-\frac{d}{ds}\right)^n \int_0^\infty e^{-st}[H(\infty)-H(t)]dt$$

$$= \sum_{j=0}^{n} \binom{n}{j} \left\{\int_0^\infty t^j e^{-st}[F(\infty)-F(t)]dt\right\} \left\{\chi(j=n) + \int_0^\infty t^{n-j} e^{-st} dH(t)\right\}$$

by (2.3.5). Thus

$$\frac{\int_0^\infty t^n e^{-st}[H(\infty)-H(t)]dt}{\int_0^\infty t^n e^{-st}[F(\infty)-F(t)]dt} = \frac{1 + \int_0^\infty e^{-st} dH(t)}{1-\omega}$$

$$+ \frac{\sum_{j=0}^{n-1} \binom{n}{j}\left\{\int_0^\infty t^j e^{-st}[F(\infty)-F(t)]dt\right\} \left\{\int_0^\infty t^{n-j} e^{-st} dH(t)\right\}}{(1-\omega) \int_0^\infty t^n e^{-st}[F(\infty)-F(t)]dt}.$$

Therefore

$$\lim_{s\to 0} \frac{\int_0^\infty t^n e^{-st}[H(\infty)-H(t)]dt}{\int_0^\infty t^n e^{-st}[F(\infty)-F(t)]dt} = \frac{1}{(1-\omega)^2}. \tag{2.3.7}$$

Hence if μ_{n+1} is the first infinite moment of F $(n = 0, 1, \ldots)$, then by (2.3.7) and Theorem 2.1,

$$\lim_{x\to\infty} \frac{\int_0^x y^n[H(\infty)-H(y)]dy}{\int_0^x y^n[F(\infty)-F(y)]dy} = \frac{1}{(1-\omega)^2} .$$

Lemma 2.6

Suppose $\int_0^\infty t^{n+1}\, dF(t) = \infty$ $(n = 0, 1, \ldots)$. Then

$$\lim_{x\to\infty} \frac{\int_0^x t^m[H(\infty)-H(t)]dt}{\int_0^x t^m[F(\infty)-F(t)]dt} = \frac{1}{(1-\omega)^2}$$

for every integer $m \geq n$.

Proof:

From the previous discussion we know that there is an integer $n_0 \leq n$ such that

$$\int_0^\infty t^{n_0+1}\, dF(t) = \infty \quad \text{and}$$

$$\lim_{s\to 0} \frac{\int_0^\infty e^{-st} t^{n_0}[H(\infty)-H(t)]dt}{\int_0^\infty e^{-st} t^{n_0}[F(\infty)-F(t)]dt} = \frac{1}{(1-\omega)^2} .$$

As a result,

$$\frac{d}{ds}\ \frac{\int_0^\infty e^{-st} t^{n_0}[H(\infty)-H(t)]dt}{\int_0^\infty e^{-st} t^{n_0}[F(\infty)-F(t)]dt} \to 0 \text{ as } s \to 0;$$

i.e.,

$$\frac{\int_0^\infty e^{-st} t^{n_0+1}[H(\infty)-H(t)]dt}{\int_0^\infty e^{-st} t^{n_0}[F(\infty)-F(t)]dt} -$$

$$\frac{\int_0^\infty e^{-st} t^{n_0}[H(\infty)-H(t)]dt \int_0^\infty e^{-st} t^{n_0+1}[F(\infty)-F(t)]dt}{\{\int_0^\infty e^{-st} t^{n_0}[F(\infty)-F(t)]dt\}^2} \to 0.$$

Consider

$$\frac{\int_0^\infty e^{-st} t^{n_0+1}[H(\infty)-H(t)]dt}{\int_0^\infty e^{-st} t^{n_0+1}[F(\infty)-F(t)]dt}$$

$$= \frac{\int_0^\infty e^{-st} t^{n_0+1}[H(\infty)-H(t)]dt}{\int_0^\infty e^{-st} t^{n_0}[F(\infty)-F(t)]dt} \cdot \frac{\int_0^\infty e^{-st} t^{n_0}[F(\infty)-F(t)]dt}{\int_0^\infty e^{-st} t^{n_0+1}[F(\infty)-F(t)]dt}$$

$$= \left\{ \frac{\int_0^\infty e^{-st} t^{n_0}[H(\infty)-H(t)]dt \int_0^\infty e^{-st} t^{n_0+1}[F(\infty)-F(t)]dt}{\{\int_0^\infty e^{-st} t^{n_o}[F(\infty)-F(t)]dt\}^2} + o(1) \right\} \times$$

$$\frac{\int_0^\infty e^{-st}t^{n_0}[F(\infty)-F(t)]dt}{\int_0^\infty e^{-st}t^{n_0+1}[F(\infty)-F(t)]dt}$$

$$= \frac{\int_0^\infty e^{-st}t^{n_0}[H(\infty)-H(t)]dt}{\int_0^\infty e^{-st}t^{n_0}[F(\infty)-F(t)]dt} + o(1)\{\frac{\int_0^\infty e^{-st}t^{n_0}[F(\infty)-F(t)]dt}{\int_0^\infty e^{-st}t^{n_0+1}[F(\infty)-F(t)dt}\}.$$

Therefore

$$\lim_{s\to 0} \frac{\int_0^\infty e^{-st}t^{n_0+1}[H(\infty)-H(t)]dt}{\int_0^\infty e^{-st}t^{n_0+1}[F(\infty)-F(t)]dt} = \frac{1}{(1-\omega)^2}.$$

By induction,

$$\lim_{s\to 0} \frac{\int_0^\infty e^{-st}t^{m}[H(\infty)-H(t)]dt}{\int_0^\infty e^{-st}t^{m}[F(\infty)-F(t)]dt} = \frac{1}{(1-\omega)^2}$$

for every integer $m \geq n$.

And now Theorem 1 ensures the result.

Lemma 2.7

$$\underline{\lim}_{x\to\infty} \frac{H(\infty)-H(x)}{F(\infty)-F(x)} \geq \frac{1}{(1-\omega)^2} .$$

Proof:

$$H(\infty)-H(x) = \sum_{n=1}^{\infty} [\omega^n - F(x)^n + F(x)^n - F^{(n)}(x)]$$

$$= \frac{\omega}{1-\omega} - \frac{F(x)}{1-F(x)} + \frac{F(x)}{1-F(x)} - H(x).$$

$$\Rightarrow \quad \frac{H(\infty)-H(x)}{F(\infty)-F(x)} \geq \frac{\frac{\omega}{1-\omega} - \frac{F(x)}{1-F(x)}}{\omega - F(x)} = \frac{\omega - F(x)}{(1-\omega)(1-F(x))(\omega-F(x))} .$$

Therefore

$$\lim_{x\to\infty} \frac{H(\infty)-H(x)}{F(\infty)-F(x)} \geq \frac{1}{(1-\omega)^2} .$$

<u>Lemma 2.8</u>

If $\mu_j = \infty$ for some $j \geq 1$, then

$$\lim_{x\to\infty} \frac{H(\infty)-H(x)}{F(\infty)-F(x)} = \frac{1}{(1-\omega)^2} .$$

Proof:

From Lemma 2.6, we have

$$\lim_{x\to\infty} \frac{\int_0^x t^{j-1}[H(\infty)-H(t)]dt}{\int_0^x t^{j-1}[F(\infty)-F(t)]dt} = \frac{1}{(1-\omega)^2} .$$

Since both the numerator and the denominator diverge, this can be true only if

$$\lim_{x\to\infty} \frac{H(\infty)-H(x)}{F(\infty)-F(x)} = \frac{1}{(1-\omega)^2} .$$

In the next chapter, we shall see that if

$$\lim_{x\to\infty} \frac{H(\infty)-H(x)}{F(\infty)-F(x)} = \frac{1}{(1-\omega)^2} ,$$

then conditioning on $\{A(t) = 1\}$ has no ultimate effect on the behavior of the renewal process. Thus this last lemma tells us that if F has an infinite moment and is well enough behaved to ensure

$$\lim_{t\to\infty} \frac{H(\infty)-H(t)}{F(\infty)-F(t)}$$

exists, then it is hopeless to try to develop a theory paralleling standard renewal theory by conditioning on $\{A(t) = 1\}$.

CHAPTER 3

THE SUB-EXPONENTIAL CASE

3.1 DEFINITIONS AND BACKGROUND

We need to know when $\lim\limits_{x\to\infty} \frac{H(\infty)-H(x)}{F(\infty)-F(x)} = \frac{1}{(1-\omega)^2}$ and the ramifications of this equality. Some preliminary definitions are required.

Let G be a proper distribution on $(0,\infty)$ such that $G(0^+)=0$ and $G(x) < 1$ for $0 < x < \infty$. G belongs to S, the sub-exponential class of distributions, if

$$\lim_{x\to\infty} \frac{1-G^{(2)}(x)}{1-G(x)} = 2. \tag{3.1.1}$$

The class S was introduced by Chistyakov (1964); he found it useful in studying branching processes.

Lemma A (Chistyakov)

If $G \in S$, then $\lim\limits_{x\to\infty} \frac{1-G(x+s)}{1-G(x)} = 1$ for all $s < \infty$.

Because 1-G(x) is a function of moderate growth,

$$1-G(x) \sim k \exp\{-\int_0^x \alpha(u)du\}$$

where $\alpha(u) \geq 0$ and $\alpha(u) \to 0$. Hence

$$e^{st}[1-G(t)] \to \infty \text{ as } t \to \infty \text{ for all } s > 0.$$

The fact that the tail of G converges to 0 more slowly than any exponential function explains the label "sub-exponential."

Lemma B (Chistyakov)

If $G \in S$, then

$$\lim_{x\to\infty} \frac{1-G^{(k)}(x)}{1-G(x)} = k. \tag{3.1.2}$$

Athreya and Ney (1972) have also used the class S to deal with branching processes and have proved the following useful lemma.

Lemma C (Athreya and Ney)

If $G \in S$, then given any $\varepsilon > 0$, there is a $D < \infty$ such that

$$\frac{1-G^{(n)}(x)}{1-G(x)} \le D(1+\varepsilon)^n$$

for all n and x.

Teugels (1975) used these lemmas to prove an important theorem for transient renewal processes.

Theorem A (Teugels)

If $F(\infty) = \omega < 1$, the following statements are equivalent.

(i) $F(\infty)^{-1}F(x) \in S$

(ii) $H(\infty)^{-1}H(x) \in S$

(iii) $\lim_{x\to\infty} \frac{H(\infty)-H(x)}{F(\infty)-F(x)} = \frac{1}{(1-\omega)^2}$.

Chistyakov's result on the expected number of particles at time t in an age-dependent branching process is now an easy consequence of Theorem A. From equation (1.4.18) we have

$$A(t) = 1-G(t) + \int_0^t [1-G(t-\tau)]dH_\alpha(\tau)$$

where H_α is the transient renewal function corresponding to the defective distribution αG. Evaluating the integral, we find

$$A(t) = 1 - (\alpha^{-1}-1)H_\alpha(t)$$

or

$$\frac{A(t)}{1-G(t)} = \frac{\alpha-(1-\alpha)H_\alpha(t)}{\alpha-\alpha G(t)} \to \frac{1}{1-\alpha} \tag{3.1.3}$$

as $t \to \infty$ if and only if $G \in S$.

We shall see that when $\omega^{-1}F(x) \in S$, the resulting transient renewal process has remarkable properties. It is worth noting, therefore, that distributions of this type can be quite simple in form. For example, $\omega^{-1}F(x) \in S$ if

(i) $\omega-F(x) \sim x^{-\alpha}L(x)$, $\alpha \geq 0$ and L is slowly varying.

(ii) $\omega-F(x) \sim k\ \exp\{-x^{\beta}\}$, $0 < \beta < 1$.

(iii) $\omega-F(x) \sim k\ \exp\ \{x(\log x)^{-\beta}\}$, $\beta > 1$.

However, if $\beta < 1$ in (iii), $\omega^{-1}F(x) \notin S$. (See Teugels (1975).)

3.2 PROPERTIES OF TRANSIENT RENEWAL PROCESSES WHEN $\omega^{-1}F(x) \in S$

Conditioning on $\{A(t) = 1\}$ focuses on only those processes alive at time t. One feels intuitively that if the process is alive at t, it should be behaving like an ordinary renewal process. Intuition is far from correct, however, when $\omega^{-1}F(x)$ is a sub-exponential distribution. In fact, many limit results are the same regardless of whether one conditions on $\{A(t) = 1\}$.

Lemma 3.1

If $\omega^{-1}F(x) \in S$, then

$$\lim_{t\to\infty} P\{N(t) = k|A(t) = 1\} = \lim_{t\to\infty} P\{N(t)=k\} = \omega^k(1-\omega).$$

Proof:

$$P\{N(t)=k,\ A(t)=1\} = \int_0^t [\omega-F(t-\tau)]dF^{(k)}(\tau) =$$

$$\omega F^{(k)}(t) - F^{(k+1)}(t). \tag{3.2.1}$$

Thus

$$P\{N(t) = k|A(t) = 1\} = \frac{\omega F^{(k)}(t)-F^{(k+1)}(t)}{(1-\omega)[H(\infty)-H(t)]}$$

$$= \frac{\omega^{k+1}-F^{(k+1)}(t)-\omega[\omega^k-F^{(k)}(t)]}{(1-\omega)[H(\infty)-H(t)]}$$

$$= \left\{\frac{\omega^{k+1}-F^{(k+1)}(t)}{\omega-F(t)} - \frac{\omega[\omega^k-F^{(k)}(t)]}{\omega-F(t)}\right\} \frac{\omega-F(t)}{(1-\omega)[H(\infty)-H(t)]}$$

$$\to \{(k+1)\omega^k - k\omega^k\}(1-\omega) = \omega^k(1-\omega), \tag{3.2.2}$$

by Lemma B and Theorem A. But

$$P\{N(t) = k\} = F^{(k)}(t) - F^{(k+1)}(t) \to \omega^k(1-\omega) \tag{3.2.3}$$

□

If $N(t)$ is the renewal count for a proper renewal process, then

$$\lim_{t\to\infty} P\{N(t) = k\} = 0 \text{ for all fixed } k.$$

Thus this simple lemma warns us that even those transient processes alive at time t do not behave like proper processes. The fact that $N(t)$ is not growing large as t increases is reflected in the asymptotic properties of the forward and backward delays and in $X_{N(t)+1}$, the lifetime spanning time t.

<u>Lemma 3.2</u>

$$P\{X_{N(t)+1} > t | A(t) = 1\} \to 1 \text{ as } t \to \infty.$$

Proof:

$$P\{X_{N(t)+1} > t, A(t) = 1\}$$

$$= P\{N(t) = 0, A(t) = 1\} + \sum_{n=1}^{\infty} P\{N(t)=n, X_{n+1}>t, A(t)=1\}$$

$$= \omega-F(t) + \sum_{n=1}^{\infty} \int_0^t [\omega-F(t)]dF^{(n)}(\tau)$$

$$=[\omega-F(t)][1+H(t)]. \tag{3.2.4}$$

Hence

$$\lim_{t\to\infty} P\{X_{N(t)+1} > t | A(t) = 1\}$$

$$= \lim_{t\to\infty} \left[\frac{\omega-F(t)}{H(\infty)-H(t)}\right]\left[\frac{1+H(t)}{1-\omega}\right] = \frac{(1-\omega)^2}{(1-\omega)^2} . \qquad (3.2.5)$$

□

Therefore, if the process is alive at some distant time t, this seems to be because one of the lifetimes begun prior to the instant t is itself longer than t. Evidently we have not observed a large number of moderate lifetimes; processes built up of many such "reasonable" X's would have died before reaching time t.

Given this result about $X_{N(t)+1} = \eta_t + \zeta_t$, it is not surprising that the forward and backward delays have degenerate limiting distributions. From (2.2.6),

$$\lim_{t\to\infty} P\{\zeta_t \le x | A(t)=1\} = \lim_{t\to\infty} \int_t^{t+x} \frac{[1-F(t+x-\tau)]dH(\tau)}{(1-\omega)[H(\infty)-H(t)]}$$

$$\le \lim_{t\to\infty} \frac{H(t+x)-H(t)}{(1-\omega)[H(\infty)-H(t)]} = 0 \text{ for all fixed x.}$$

Similarly, (2.1.1) yields

$$\lim_{t\to\infty} P\{\eta_t \le x | A(t)=1\} = \lim_{t\to\infty} \int_{t-x}^{t} \frac{[\omega-F(t-\tau)]dH(\tau)}{(1-\omega)[H(\infty)-H(t)]}$$

$$\le \frac{\omega}{1-\omega} \lim_{t\to\infty} \frac{H(t)-H(t-x)}{H(\infty)-H(t)} = 0 \text{ for all fixed x.}$$

Before studying ζ_t and η_t more closely, we prove a simple lemma.

Lemma 3.3

Suppose $G \in S$. Then for any $\varepsilon > 0$ and integer k, there exists a $\Delta(\varepsilon,k)$ such that

$$\lim_{t\to\infty} \int_{\Delta}^{t} [\frac{1-G(t-\tau)]}{1-G(t)} \, dG^{(k)}(\tau) < \varepsilon \; . \tag{3.2.6}$$

Proof:

Choose $\Delta(\varepsilon,k)$ such that $G^{(k)}(\Delta) > 1-\varepsilon$. We know

$$\lim_{t\to\infty} \frac{G^{(k)}(t)-G^{(k+1)}(t)}{1-G(t)} = 1 = \lim_{t\to\infty} \int_0^t \frac{[1-G(t-\tau)]}{1-G(t)} \, dG^{(k)}(\tau) \tag{3.2.7}$$

and

$$\int_0^{\Delta} \frac{[1-G(t-\tau)]}{1-G(t)} \, dG^{(k)}(\tau) \geq G^{(k)}(\Delta) > 1-\varepsilon. \qquad \square$$

Lemma 3.4

Suppose $\omega^{-1}F(x) \in S$. Then

$$\lim_{t\to\infty} P\{\eta_t \leq t-c \,|\, A(t)=1,\ N(t)\geq 1\} = \frac{H(\infty)-H(c)}{H(\infty)} \; .$$

Proof:

$$P\{\eta_t \leq t-c,\ A(t)=1\} = \int_c^t [\omega-F(t-\tau)]dH(\tau) \tag{3.2.8}$$

while

$$P\{A(t)=1,\ N(t)\geq 1\} = \int_0^t [\omega-F(t-\tau)]dH(\tau) = F(t)-(1-\omega)H(t). \tag{3.2.9}$$

Note that

$$\lim_{t\to\infty} \frac{\omega-(1-\omega)H(t)}{F(t)-(1-\omega)H(t)} = \left\{\frac{\omega-(1-\omega)H(t)-[\omega-F(t)]}{\omega-(1-\omega)H(t)}\right\}^{-1}$$

$$= \{1 - (1-\omega)\}^{-1} = \omega^{-1}. \tag{3.2.10}$$

Thus

$$P\{\eta_t \le t-c \,|\, A(t)=1,\ N(t)\ge 1\}$$

$$= \int_c^t \frac{[\omega-F(t-\tau)]dH(t)}{(1-\omega)[H(\infty)-H(t)]} \cdot \frac{\omega - (1-\omega)H(t)}{F(t)-(1-\omega)H(t)}\ . \tag{3.2.11}$$

Choose $\varepsilon > 0$. Then there is some $\Delta(\varepsilon) > c$ such that

$$\int_\Delta^t \frac{\omega-F(t-\tau)}{(1-\omega)[H(\infty)-H(t)]}\, dH(\tau) < \varepsilon$$

by Lemma 3.3. Hence

$$\frac{[\omega-F(t-c)][H(\Delta)-H(c)]}{(1-\omega)[H(\infty)-H(t)]} \le \int_c^t \frac{[\omega-F(t-\tau)]dH(\tau)}{(1-\omega)[H(\infty)-H(t)]}$$

$$\le \frac{[\omega-F(t-\Delta)][H(\Delta)-H(c)]}{(1-\omega)[H(\infty)-H(t)]} + \varepsilon. \tag{3.2.12}$$

But

$$\lim_{t\to\infty} \frac{\omega-F(t-s)}{(1-\omega)[H(\infty)-H(t)]} = (1-\omega)$$

for all fixed s and ε is arbitrarily small while Δ is as large as

we like. Therefore

$$\lim_{t\to\infty} \int_c^t \frac{[\omega-F(t-\tau)]dH(\tau)}{(1-\omega)[H(\infty)-H(t)} = (1-\omega)[H(\infty)-H(c)]. \tag{3.2.13}$$

The conclusion follows from (3.2.10), (3.2.11), and (3.2.13).

□

Lemma 3.5

If $\omega-F(x) \sim x^{-\alpha}L(x)$, then

$P\{\zeta_t \le \lambda t | A(t)=1\} \to 1-(1+\lambda)^{-\alpha}$ for $\lambda > 0$.

Proof:

By Theorem A, $H(\infty)-H(x) \sim \dfrac{x^{-\alpha}L(x)}{(1-\omega)^2}$

$$P\{\zeta_t \le \lambda t | A(t)=1\} = \int_t^{t(1+\lambda)} \frac{[1-F(t+\lambda t-y]dH(y)}{(1-\omega)[H(\infty)-H(t)]}$$

$$= \frac{H(t+\lambda t)-H(t)}{H(\infty)-H(t)} + \int_t^{t(1+\lambda)} \frac{(\omega-F(t+\lambda t-y)]dH(y)}{(1-\omega)[H(\infty)-H(t)]}. \tag{3.2.14}$$

Now

$$\lim_{t\to\infty} \frac{H(t+\lambda t)-H(t)}{H(\infty)-H(t)} = 1 - \lim_{t\to\infty} \frac{H(\infty)-H(t+\lambda t)}{H(\infty)-H(t)} = 1-(1+\lambda)^{-\alpha} \tag{3.2.15}$$

while

$$\lim_{t\to\infty} \int_t^{t(1+\lambda)} \frac{[\omega-F(t+\lambda t-y)]dH(y)}{(1-\omega)[H(\infty)-H(t)]}$$

$$\leq \lim_{t\to\infty} \frac{H(\infty)-H(t+\lambda t)}{(1-\omega)[H(\infty)-H(t)]} \int_t^{t+\lambda t} \frac{H(\infty)-H(t+\lambda t-y)dH(y)}{H(\infty)-H(t+\lambda t)}$$

$$= 0 \text{ by Lemma 3.3.} \tag{3.2.16}$$

□

To study the moments of N(t), we begin by finding expressions for these moments. The factorial polynomial $x_{(n)}$ is defined as $x_{(n)} = x(x-1)\cdots(x-n+1)$; $x_{(o)} \equiv 1$ (n=0, 1, 2, ...). Stirling's numbers of the second kind are used to express x^n as a linear combination of factorial powers of x no higher than the nth.

$$x^n = t_{n1}x_{(1)} + t_{n2}x_{(2)} + \cdots + t_{nn}x_{(n)}.$$

Lemma 3.6

a) $EN(t)^k A(t) = \sum_{j=1}^{k} j!t_{kj}[\omega H^{(j)}(t)-(1-\omega)H^{(j+1)}(t)]$ and

b) $EN(t)^k = \sum_{j=1}^{k} j!t_{kj}H^{(j)}(t)$

where the t_{kj}'s are Stirling's numbers of the second kind.

Proof:

a) $EN(t)^k A(t) = \sum_{n=1}^{\infty} n^k \int_0^t [\omega-F(t-\tau)]dF^{(n)}(\tau)$

$$= \sum_{n=1}^{\infty} n^k[\omega F^{(n)}(t)-F^{(n+1)}(t)] \tag{3.2.17}$$

$$\Rightarrow L\{EN(t)^k A(t)\} = [\omega-F^*(s)]\sum_{n=1}^{\infty} n^k F^*(s)^n. \tag{3.2.18}$$

$$n^k = \sum_{j=1}^{k} t_{kj} n_{(j)}, \text{ so we can write}$$

$$L\{EN(t)^k A(t)\} = [\omega - F^*(s)] \sum_{j=1}^{k} t_{kj} \sum_{n=j}^{\infty} n_{(j)} F^*(s)^n$$

$$= [\omega - F^*(s)] \sum_{j=1}^{k} t_{kj} F^*(s)^j \sum_{n=j}^{\infty} n_{(j)} F^*(s)^{n-j}$$

$$= [\omega - F^*(s)] \sum_{j=1}^{k} j! t_{kj} F^*(s)^j [1 - F^*(s)]^{j+1}$$

$$= \sum_{j=1}^{k} j! t_{kj} [\omega - F^*(s)] H^*(s)^j [1 - F^*(s)]^{-1}. \qquad (3.2.19)$$

From the equation $H^*(s) = F^*(s)[1+H^*(s)]$, we see that $(1-F^*(s))(1+H^*(s)) = 1$. Hence

$$[\omega - F^*(s)][1-F^*(s)]^{-1} = \omega - (1-\omega)H^*(s). \qquad (3.2.20)$$

Substituting the expression in (3.2.20) into (3.2.19), we find

$$L\{EN(t)^k A(t)\} = \sum_{j=1}^{k} j! t_{jk} \{\omega H^*(s)^j - (1-\omega) H^*(s)^{j+1}\}$$

and hence

$$EN(t)^k A(t) = \sum_{j=1}^{k} j! t_{kj} \{\omega H^{(j)}(t) - (1-\omega) H^{(j+1)}(t)\}. \qquad (3.2.21)$$

b) $$EN(t)^k = \sum_{n=1}^{\infty} n^k \int_0^t [1-F(t-\tau)]dF^{(n)}(\tau)$$

$$= \sum_{n=1}^{\infty} n^k [F^{(n)}(t)-F^{(n+1)}(t)] \tag{3.2.22}$$

Therefore

$$L\{EN(t)^k\} = [1-F^*(s)] \sum_{n=1}^{\infty} n^k F^*(s)^n$$

$$= \sum_{j=1}^{k} j!t_{kj}H^*(s)^j \text{ from } (3.2.19).$$

Hence

$$EN(t)^k = \sum_{j=1}^{k} j!t_{kj}H^{(j)}(t). \tag{3.2.23}$$

□

<u>Theorem 3.1</u>

$\lim\limits_{t\to\infty} E[N(t)^k|A(t) = 1] = \lim\limits_{t\to\infty} EN(t)^k = \sum_{j=1}^{k} j!t_{kj}[\frac{\omega}{1-\omega}]^j$ if and only if $\omega^{-1}F(x) \in S$.

Proof:

(i) Suppose $\omega^{-1}F(x) \in S$. Then $H(\infty)^{-1}H(x) \in S$ and by Lemma B,

$$\frac{[\frac{\omega}{1-\omega}]^n - H^{(n)}(t)}{H(\infty)-H(t)} \to n(\frac{\omega}{1-\omega})^{n-1} \qquad n=1, 2, \ldots \tag{3.2.24}$$

Thus

$$\frac{H^{(n)}(t)}{H(\infty)-H(t)} = \frac{(1-\omega)H^{(n)}(t)}{q(t)} = \frac{\omega^n}{(1-\omega)^{n-1}q(t)} - \frac{n\omega^{n-1}}{(1-\omega)^{n-1}} + o(1). \tag{3.2.25}$$

Multiplying by $\frac{\omega}{1-\omega}$, we have

$$\frac{\omega H^{(n)}(t)}{q(t)} = \frac{\omega^{n+1}}{(1-\omega)^n q(t)} - \frac{n\omega^n}{(1-\omega)^n} + o(1). \tag{3.2.26}$$

Taking n = k in (3.2.26) and n = k+1 in (3.2.25) and subtracting, we find

$$\frac{\omega H^{(k)}(t)-(1-\omega)H^{(k+1)}(t)}{q(t)} = \frac{\omega^k}{(1-\omega)^k} + o(1). \tag{3.2.27}$$

Therefore

$$\lim_{t\to\infty} E[N(t)^k|A(t)=1] = \lim_{t\to\infty} \sum_{j=1}^{k} j!t_{kj}\left[\frac{\omega H^{(j)}(t)-(1-\omega)H^{(j+1)}(t)}{q(t)}\right]$$

$$= \sum_{j=1}^{k} j!t_{kj}\left(\frac{\omega}{1-\omega}\right)^j = \lim_{t\to\infty} \sum_{j=1}^{k} j!t_{kj}H^{(j)}(t)$$

$$= \lim_{t\to\infty} EN(t)^k \qquad \text{by (3.2.21) and (3.2.23).}$$

(ii) Now suppose

$$\lim_{t\to\infty} E[N(t)^k|A(t)=1] = \sum_{j=1}^{k} j!t_{kj}\left(\frac{\omega}{1-\omega}\right)^j.$$

Taking k = 1, we have

$$\int_0^t \frac{[H(\infty)-H(t-\tau)]dH(\tau)}{H(\infty)-H(t)} \to \frac{\omega}{1-\omega} \qquad \text{by (2.1.2).}$$

That is,

$$\frac{H(\infty)^2-H^{(2)}(t)}{H(\infty)-H(t)} - \frac{H(\infty)^2-H(\infty)H(t)}{H(\infty)-H(t)} \to \frac{\omega}{1-\omega} .$$

But then

$$\frac{H(\infty)^2-H^{(2)}(t)}{H(\infty)-H(t)} \to \frac{2\omega}{1-\omega} ,$$

which is equivalent to saying $H(\infty)^{-1}H(t) \in S$. Theorem A implies $\omega^{-1}F(t) \in S$ as well. □

Smith (1959) has shown that when $\{X_i\}$ is a proper renewal process $E[N(t)^k]$ is asymptotically a k<u>th</u> degree polynomial in t. However, when $\{X_i\}$ is transient and $\omega^{-1}F(x) \in S$, the moments of N(t) converge to finite values, even when we restrict our attention to moments of processes alive at t. This is an important discrepancy. For example, many of the standard renewal theoretic results cited in Chapter 1 are direct outgrowths of the Key Renewal Theorem, which depends in turn on the approximate linearity of $H(t) = EN(t)$. Our efforts to recover these standard results in the transient setting by conditioning on $\{A(t) = 1\}$ are completely fruitless when $\omega^{-1}F(x) \in S$ because $E[N(t)|A(t)=1] \to \frac{\omega}{1-\omega}$.

In the sub-exponential case, conditioning on $\{A(t) = 1\}$ seems to have little effect on the long term properties of the transient process. We have seen, for example, that

$$\lim_{t\to\infty}(P\{N(t)=k|A(t)=1\} - P\{N(t)=k\}) = 0$$

and

$$\lim_{t\to\infty}(E[N(t)^k|A(t)=1] - E[N(t)^k]) = 0.$$

This is particularly surprising in light of later chapters, where we shall see that when the tail of F decreases exponentially, the conditioning scheme is very effective in developing a theory paralleling the standard renewal theory.

CHAPTER 4

THE EXPONENTIAL CASE

4.1 THE PROPER DISTRIBUTION $\tilde{F}$

Suppose

$$\lim_{t\to\infty} \frac{q(t+s)}{q(t)} = \psi(s) = e^{-\sigma s}, \ \sigma > 0.$$

Then, as we recall from Chapter 2, $e^{\sigma t}[H(\infty)-H(t)]$ is a function of moderate growth and $\int_0^\infty e^{ct}[H(\infty)-H(t)]dt < \infty$ for all $c < \sigma$. The tail of H (and of F) now falls off exponentially, in contrast with the sub-exponential case examined in the last chapter.

We can take $-s = c < \sigma$ in (2.3.5) and conclude

$$(1-\omega)\int_0^\infty e^{ct}[H(\infty)-H(t)]dt = \{\int_0^\infty e^{ct}[F(\infty)-F(t)]dt\}\{1+\int_0^\infty e^{ct}dH(t)\}$$

or

$$\int_0^\infty e^{ct}[F(\infty)-F(t)]dt = \frac{(1-\omega)\int_0^\infty e^{ct}[H(\infty)-H(t)]dt}{(1-\omega)^{-1} + c\int_0^\infty e^{ct}[H(\infty)-H(t)]dt} . \quad (4.1.1)$$

Now either

a) $\lim_{c\uparrow\sigma} \int_0^\infty e^{ct}[H(\infty)-H(t)]dt = \ell < \infty$

or

b) $\lim_{c\uparrow\sigma} \int_0^\infty e^{ct}[H(\infty)-H(t)]dt = \infty.$

When a) holds,

$$\lim_{c\uparrow\sigma} \int_0^\infty e^{ct}[F(\infty)-F(t)]dt = \frac{(1-\omega)\ell}{(1-\omega)^{-1}+\sigma\ell}$$

$$= \int_0^\infty e^{\sigma t}[F(\infty)-F(t)]dt \tag{4.1.2}$$

by the Monotone Convergence Theorem.

However, if the integral diverges and b) obtains, then

$$\lim_{c\uparrow\sigma} \int_0^\infty e^{ct}[F(\infty)-F(t)]dt = \frac{(1-\omega)}{\sigma}$$

$$= \int_0^\infty e^{\sigma t}[F(\infty)-F(t)]dt. \tag{4.1.3}$$

Of course

$$\int_0^\infty e^{\sigma t}[F(\infty)-F(t)]dt = \frac{F^*(-\sigma)-\omega}{\sigma}$$

and hence either

a) $F^*(-\sigma) = \dfrac{1-\omega}{1+[\sigma(1-\omega)\ell]^{-1}} + \omega<1$

or

b) $F^*(-\sigma) = 1.$

Lemma 4.1

If $q(t) \sim ke^{-\sigma t}$, then $F^*(-\sigma) = 1$.

Proof:

If $q(t) = (1-\omega)[H(\infty)-H(t)] \sim ke^{-\sigma t}$,

$$\lim_{c \uparrow \sigma} \int_0^\infty e^{ct}[H(\infty)-H(t)]dt = \infty$$

and the result follows from the previous discussion. □

Suppose that $F^*(-\sigma) = 1$. Then we can define the proper distribution $\tilde{F}$ by

$$d\tilde{F}(x) = e^{\sigma x} dF(x). \tag{4.1.3}$$

Write $\tilde{E}$ and $\tilde{P}$ for expectations and probabilities with respect to this distribution; let $\tilde{\mu}_k = \tilde{E}X^k \leq \infty$.

We say that a stochastic process $G(t)$ is *natural* if its distribution depends only on t and $X_1, X_2, \ldots, X_{N(t)+1}$. Cumulative processes are therefore natural. $G(t)$ is a proper process defined for all t if $X_1 \sim \tilde{F}$; it is transient if $X_1 \sim F$. The proper and transient processes are said to be *equivalent* and their expected values are related.

$$\tilde{E}e^{-\sigma S_{N(t)+1}} G(t) = \sum_{n=0}^{\infty} \int_{\{N(t)=n\}} G(t,x_1,\ldots,x_{n+1})$$

$$e^{-\sigma(x_1+\cdots+x_{n+1})} d\tilde{F}(x_1)\cdots d\tilde{F}(x_{n+1})$$

$$= \sum_{n=0}^{\infty} \int_{\{N(t)=n\}} G(t,x_1,\cdots,x_{n+1})dF(x_1)\cdots dF(x_{n+1}). \tag{4.1.4}$$

In the right hand side of (4.1.4), G(t) is a transient process. However, the expression is well-defined because on each set $\{N(t)=n\}$, $X_1+\cdots+X_n \le t < \infty$ and $X_{n+1} < \infty$ since we are integrating with respect to $dF(x_{n+1})$. Thus the process G is alive at time t.

Remembering that $S_{N(t)+1} = t+\zeta_t$, we can rewrite (4.1.4) as

$$e^{-\sigma t}\tilde{E}e^{-\sigma\zeta_t}G(t) = EG(t)A(t). \tag{4.1.5}$$

<u>Lemma 4.2</u>

If $F^*(-\sigma) = 1$ and $\tilde{\mu}_1 = \int_0^\infty xe^{\sigma x}dF(x) < \infty$, then $q(t) \sim \frac{(1-\omega)e^{-\sigma t}}{\sigma\tilde{\mu}_1}$.

Proof:

Let $G(t) \equiv 1$ in (4.1.5). Then

$$e^{-\sigma t}\tilde{E}e^{-\sigma\zeta_t} = q(t). \tag{4.1.6}$$

From (1.1.13),

$$\lim_{t\to\infty} \tilde{E}e^{-\sigma\zeta_t} = \frac{1-\omega}{\sigma\tilde{\mu}_1} = \tilde{K}^*(\sigma). \qquad \Box$$

We shall assume that $F^*(-\sigma)=1$ throughout the remainder of this chapter and in the next two chapters. Equations (4.1.5) and (4.1.6) yield

$$E[G(t)|A(t)=1] = \frac{\tilde{E}e^{-\sigma\zeta_t}G(t)}{\tilde{E}e^{-\sigma\zeta_t}}. \tag{4.1.7}$$

Suppose G(t) is a cumulative process. Then the right hand side of (4.1.7) involves G(t), a proper process built up from N(t) independent random variables and some extra piece unlikely to affect the distribution of G(t) substantively for large t, and ζ_t, the time from t until the next event. In a loose sense, ζ_t is determined by local X's around time t while G(t) depends on all tours up to time t. We are led to consider the circumstances when

$$\frac{\tilde{E}e^{-\sigma\zeta_t}G(t)}{\tilde{E}e^{-\sigma\zeta_t}} - \tilde{E}G(t) \to 0 \text{ as } t \to \infty \tag{4.1.8}$$

for when (4.1.8) holds, we have

$$E[G(t)|A(t)=1] - \tilde{E}G(t) \to 0 \text{ as } t \to \infty \tag{4.1.9}$$

and we can apply our considerable knowledge about $\tilde{E}G(t)$ to the transient process.

4.2 THE ASYMPTOTIC INDEPENDENCE OF ζ_t AND G(t)

The notions of sluggish events and processes are helpful in finding conditions guaranteeing that (4.1.9) holds. An event A(t) is *sluggish* if $\tilde{P}\{\chi(A(t)) \neq \chi(A(t+T))\} \to 0$ as $t \to \infty$ for all fixed $T > 0$. A(t) is natural if $\chi(A(t))$ is a natural process. The following unpublished theorem is a major step toward the desired conditions assuring the asymptotic independence of G(t) and ζ_t.

Theorem B (Smith)

Suppose A(t) is a sluggish and natural event. Then if $\tilde{\mu}_1 < \infty$,

$$\tilde{P}\{\zeta_t \le x, A(t)\} - \tilde{K}(x)\tilde{P}\{A(t)\} \to 0 \text{ as } t \to \infty. \tag{4.2.1}$$

Proof:

Let $G_{A_t}(y) = \tilde{P}\{\zeta_t \le y, A(t)\}$ and choose $\varepsilon > 0$.

$$\tilde{P}\{\zeta_{t+\tau} \le x, A(t)\} = \tilde{P}\{\zeta_{t+\tau} \le x, S_{N(t)+1} \le t+\tau, A(t)\}$$

$$+ \tilde{P}\{\zeta_{t+\tau} \le x, S_{N(t)+1} > t+\tau, A(t)\}$$

$$= \tilde{P}\{\zeta_{t+\tau} \le x, \zeta_t \le \tau, A(t)\} + \tilde{P}\{\tau < \zeta_t \le \tau + x, A(t)\}$$

$$= \int_0^\tau \tilde{P}\{\zeta_{\tau-y} \le x\} dG_{A_t}(y) + \int_\tau^{\tau+x} dG_{A_t}(y) \tag{4.2.2}$$

We know that when $\tilde{\mu}_1 < \infty$, $\lim_{t\to\infty} \tilde{P}\{\zeta_t \le x\} = \tilde{K}(x)$, a continuous distribution function. Thus the convergence is uniform with respect to x. Hence there is some $T(\varepsilon)$ such that

$$|\tilde{P}\{\zeta_t \le x\} - \tilde{K}(x)| < \varepsilon \text{ for all } x$$

whenever $t \ge T$. Let $\tau = 2T$ in (4.2.2). Then

$$\tilde{P}\{\zeta_{t+2T} \le x, A(t)\} = \int_0^{2T} \tilde{P}\{\zeta_{2T-y} \le x\} dG_{A_t}(y) + \int_{2T}^{2T+x} dG_{A_t}(y) \tag{4.2.3}$$

$$\le \int_0^T \tilde{P}\{\zeta_{2T-y} \le x\} dG_{A_t}(y) + \int_T^\infty dG_{A_t}(y)$$

$$\le [\tilde{K}(x) + \varepsilon]\tilde{P}\{A(t)\} + \tilde{P}\{\zeta_t > T\}$$

$$\leq [\tilde{K}(x)+\varepsilon]\tilde{P}\{A(t)\} + 1-\tilde{K}(T)+\varepsilon.$$

But for sufficiently large T, $1-\tilde{K}(T) < \varepsilon$ and thus

$$\tilde{P}\{\zeta_{t+2T}\leq x,A(t)\} \leq \tilde{K}(x)\tilde{P}\{A(t)\} + 3\varepsilon. \tag{4.2.4}$$

(4.2.3) also yields

$$\tilde{P}\{\zeta_{t+2T}\leq x,A(t)\} \geq [\tilde{K}(x)-\varepsilon]G_{A_t}(T) = [\tilde{K}(x)-\varepsilon][\tilde{P}\{A(t)\} -\tilde{P}\{A(t),\zeta_t>T\}]$$

$$\geq \tilde{K}(x)\tilde{P}\{A(t)\} - \tilde{P}\{A(t),\zeta_t>T\}-\varepsilon$$

$$\geq \tilde{K}(x)\tilde{P}\{A(t)\} - [1-\tilde{K}(T)+\varepsilon]-\varepsilon$$

$$\geq \tilde{K}(x)\tilde{P}\{A(t)\}-3\varepsilon. \tag{4.2.5}$$

Therefore

$$\overline{\lim_{t\to\infty}}\left|\tilde{P}\{\zeta_{t+2T}\leq x,A(t)\}-\tilde{K}(x)\tilde{P}\{A(t)\}\right|<3\varepsilon(T) \tag{4.2.6}$$

where $\varepsilon(T)\to 0$ as $T\to\infty$. Because A(t) is sluggish,

$$\tilde{P}\{\zeta_{t+2T}\leq x,A(t)\}-\tilde{P}\{\zeta_{t+2T}\leq x,A(t+2T)\} \to 0$$

and

$$\tilde{P}\{A(t)\} - \tilde{P}\{A(t+2T)\} \to 0.$$

Therefore

$$\overline{\lim_{t\to\infty}}\left|\tilde{P}\{\zeta_{t+2T}\le x, A(t+2T)\}-\tilde{K}(x)\tilde{P}\{A(t+2T)\}\right|$$

$$= \overline{\lim_{t\to\infty}}\left|\tilde{P}\{\zeta_{t+2T}\le x, A(t)\}-\tilde{}(x)\tilde{P}\{A(t)\}\right| < 3\varepsilon(T).$$

That is,

$$\tilde{P}\{\zeta_t\le x, A(t)\}-\tilde{K}(x)\tilde{P}\{A(t)\} \to 0 \text{ as } t \to \infty. \qquad \square$$

<u>Corollary 4.1</u>

If $A(t)$ is a sluggish and natural event and $\tilde{\mu}_1 < \infty$, then

$$\frac{\tilde{E}e^{-\sigma\zeta_t}\chi(A(t))}{\tilde{E}e^{-\sigma\zeta_t}} - \tilde{P}\{A(t)\} \to 0. \tag{4.2.7}$$

Proof:

Choose $\varepsilon>0$. There exists an N such that if $\zeta_t\ge N$, $e^{-\sigma\zeta_t}<\varepsilon$. Let $B_N = \{\zeta_t<N\}$.

$$\tilde{E}e^{-\sigma\zeta_t}\chi(A(t))=\tilde{E}e^{-\sigma\zeta_t}\chi(A(t)\cap B_N) + \tilde{E}e^{-\sigma\zeta_t}\chi(A(t)\cap B_N{}^c) \tag{4.2.8}$$

$$\tilde{E}e^{-\sigma\zeta_t}\chi(A(t)\cap B_N{}^c) < \varepsilon. \tag{4.2.9}$$

Establish a partition $0 = x_0<x_1<\cdots<x_n=N$ with mesh m.

By Theorem B

$$\underline{\lim_{t\to\infty}} \{\tilde{E}e^{-\sigma\zeta_t}\chi(A(t)\cap B_N) - \sum_{j=0}^{n-1} e^{-\sigma x_{j+1}}\tilde{P}\{A(t)\}[\tilde{K}(x_{j+1})-\tilde{K}(x_j)]\} \geq 0 \tag{4.2.10}$$

$$\underline{\lim_{t\to\infty}} \sum_{j=0}^{n-1} e^{-\sigma x_j}\tilde{P}\{A(t)\}[\tilde{K}(x_{j+1})-\tilde{K}(x_j)] - \tilde{E}e^{-\sigma\zeta_t}\chi(A(t)\cap B_N)\} \geq 0. \tag{4.2.11}$$

Letting $m \to 0$, we conclude

$$\lim_{t\to\infty} \{\tilde{E}e^{-\sigma\zeta_t}\chi(A(t)\cap B_N) - \tilde{P}\{A(t)\} \int_0^N e^{-\sigma x} d\tilde{K}(x)\} = 0. \tag{4.2.12}$$

(4.2.8), (4.2.9), (4.2.12), and the arbitrary size of ε imply

$$\tilde{E}e^{-\sigma\zeta_t}\chi(A(t)) - \tilde{P}\{A(t)\}\tilde{K}^*(\sigma) \to 0.$$

The conclusion follows because when $\tilde{\mu}_1 < \infty$, $\tilde{E}e^{-\sigma\zeta_t} \to \tilde{K}^*(\sigma)$. A direct application of this corollary produces the following lemma.

Lemma 4.3

Let $W(t)$ be a cumulative process such that $\tilde{E}Y_1^* = \tilde{\kappa}_1^* < \infty$ and $\tilde{\mu}_1 < \infty$. Then

$$P\{ \left|\frac{W(t)}{t} - \frac{\tilde{\kappa}_1}{\tilde{\mu}_1}\right| > \varepsilon \,\Big|\, A(t) = 1\} \to 0 \text{ as } t \to \infty. \tag{4.2.13}$$

Proof:

Let $A(t) = \{\left|\frac{W(t)}{t} - \frac{\tilde{\kappa}_1}{\tilde{\mu}_1}\right| > \varepsilon\}$.

$A(t)$ is sluggish because $\tilde{P}\{\chi(A(t)) \neq \chi(A(t+T))\} \leq \tilde{P}\{A(t)\} + \tilde{P}\{A(t+T\} \to 0$ as $t \to \infty$ since $\frac{W(t)}{t} \to \frac{\tilde{\kappa}_1}{\tilde{\mu}_1}$ a.s.,($\tilde{P}$) when $\tilde{\mu}_1 < \infty$ and $\tilde{\kappa}_1{}^* < \infty$.

$$P\left\{\left|\frac{W(t)}{t} - \frac{\tilde{\kappa}_1}{\tilde{\mu}_1}\right| > \varepsilon \;\middle|\; A(t)=1\right\} = \frac{\tilde{E}e^{-\sigma\zeta_t}\chi(A(t))}{\tilde{E}e^{-\sigma\zeta t}} \tag{4.2.14}$$

$$= \tilde{P}(A(t)) + o(1) \text{ by Corollary 4.1.} \qquad \square$$

To translate the asymptotic independence of ζ_t and $A(t)$ into an independence result for ζ_t and some process $G(t)$, we define sluggish processes. A real valued process $G(t)$ is sluggish if for all x outside a set E of Lebesgue measure 0 and for all fixed $T > 0$

$$\tilde{P}\{G(t) \leq x < G(t+T\} + \tilde{P}\{G(t) > x \geq G(t+T)\} \to 0.$$

<u>Lemma 4.4</u>

If $G(t)$ has a limiting proper distribution, then $G(t)$ is sluggish if and only if $G(t+T) - G(t) \overset{\tilde{P}}{\to} 0$ as $t \to \infty$.

Proof:

(i) Suppose $G(t)$ is sluggish and has limiting distribution J. Fix $\varepsilon > 0$. There exists some $N(\varepsilon)$ such that $\pm N$ are continuity points of J and $J(N) - J(-N) > 1-\varepsilon$. Let $A_N = (-N,N]$.

$$\tilde{P}\{|G(t+T)-G(t)|>\varepsilon\} \leq \tilde{P}\{|G(t+T-G(t)|>\varepsilon,\ G(t+T)\in A_N,$$

$$G(t)\in A_N\} + \tilde{P}\{G(t+T)\in A_N{}^c\} + \tilde{P}\{G(t)\in A_N{}^c\}. \tag{4.2.15}$$

Let $-N = x_0 < x_1 < \cdots < x_M = N$ be a partition of A_N such that $x_{j+1} - x_j < \frac{\varepsilon}{2}$ and $x_j \in E^c$

$$\tilde{P}\{|G(t+T-G(t)|>\varepsilon,\ G(t+T)\in A_N,\ G(t)\in A_N\}$$

$$\leq \sum_{j=0}^{M-1} \tilde{P}\{G(t+T)\in(x_j,x_{j+1}],\ G(t) \notin (x_j,x_{j+1}]\}$$

$$\leq \sum_{j=0}^{M-1} \tilde{P}\{G(t)\leq x_j<G(t+T)\} + \sum_{j=0}^{M-1} \tilde{P}\{G(t+T)\leq x_j<G(t)\}$$

$$\to 0 \text{ as } t \to \infty \text{ since } G \text{ is sluggish.} \tag{4.2.16}$$

$$\overline{\lim_{t\to\infty}}\ [\tilde{P}\{G(t+T)\in A_N{}^c\} + \tilde{P}\{G(t)\in A_N{}^c\}] \leq 2\varepsilon. \tag{4.2.17}$$

ε is arbitrary; thus (4.2.15), (4.2.16), and (4.2.17) imply

$$G(t+T) - G(t) \overset{\tilde{P}}{\to} 0.$$

(ii) Now suppose $G(t)$ has limiting distribution J and $G(t+T) - G(t) \overset{\tilde{P}}{\to} 0$. Define $E = \{x\colon J(x) \neq J(x-)\}$; E has Lebesgue measure 0. Let $x\in E^c$ and choose $\varepsilon > 0$.

$$\tilde{P}\{G(t)\leq x<G(t+T)\} = \tilde{P}\{G(t)\leq x\} - \tilde{P}\{G(t)\leq x,\ G(t+T)\leq x\}$$

$$\leq \tilde{P}\{G(t)\leq x\} - \tilde{P}\{G(t)\leq x-\varepsilon, |G(t+T)-G(t)|<\varepsilon\}$$

$$= \tilde{P}\{G(t)\leq x\} - \tilde{P}\{G(t)\leq x-\varepsilon\} + \tilde{P}\{G(t)\leq x-\varepsilon, |G(t+T)-G(t)|\geq\varepsilon\}$$

$$\leq \tilde{P}\{G(t)\leq x\} - \tilde{P}\{G(t)\leq x-\varepsilon\} + \tilde{P}\{|G(t+T)-G(t)|\geq\varepsilon\} \tag{4.2.18}$$

$\to J(x) - J(x-\varepsilon)$ by assumption.

But x is a continuity point of J and ε is arbitrary. Thus $\tilde{P}\{G(t)\le x<G(t+T)\} \to 0$ as $t \to \infty$; the argument for $\tilde{P}\{G(t)>x\ge G(t+T)\}$ is exactly the same. □

Lemma 4.5

Suppose $G(t)$ is a sluggish and natural process such that for every $\varepsilon > 0$ there is a $\Delta(\varepsilon)$ making

$$\tilde{E}G(t)\chi(|G(t)| \ge \Delta) < \varepsilon$$

for all sufficiently large t. If $\tilde{\mu}_1 > \infty$, then

$$\frac{\tilde{E}e^{-\sigma\zeta_t}G(t)}{\tilde{E}e^{-\sigma\zeta_t}} - \tilde{E}G(t) \to 0 \text{ as } t \to \infty. \tag{4.2.19}$$

Proof:

Fix ε and choose a concomitant Δ. Define $B_\Delta = (-\Delta,\Delta]$.

$$\tilde{E}e^{-\sigma\zeta_t}G(t) = \tilde{E}e^{-\sigma\zeta_t}G(t)\chi(G(t)\in B_\Delta) + \tilde{E}e^{-\sigma\zeta_t}G(t)\chi(G(t)\in B_\Delta{}^c)$$

$$\overline{\lim_{t\to\infty}}\ \frac{|\tilde{E}e^{-\sigma\zeta_t}G(t)\chi(G(t)\in B_\Delta{}^c)|}{\tilde{E}e^{-\sigma\zeta_t}} \le \varepsilon\tilde{K}^*(\sigma)^{-1}. \tag{4.2.20}$$

Now establish a partition $-\Delta = x_0<x_1<\cdots<x_N = \Delta$ having mesh m.

$$\tilde{E}e^{-\sigma\zeta_t}G(t)\chi(G(t)\in B_\Delta) = \sum_{n=0}^{N-1} \tilde{E}e^{-\sigma\zeta_t}G(t)\chi(x_n<G(t)\le x_{n+1})$$

and hence

$$\sum_{n=0}^{N-1} \frac{x_n \tilde{E} e^{-\sigma\zeta_t} \chi(x_n < G(t) \le x_{n+1})}{\tilde{E} e^{-\sigma\zeta_t}} \le \frac{\tilde{E} e^{-\sigma\zeta_t} G(t)\chi(G(t) \in B_\Delta)}{\tilde{E} e^{-\sigma\zeta_t}}$$

$$\le \sum_{n=0}^{N-1} \frac{x_{n+1} \tilde{E} e^{-\sigma\zeta_t} \chi(x_n < G(t) \le x_{n+1})}{\tilde{E} e^{-\sigma\zeta_t}} . \qquad (4.2.21)$$

Because the events $A_n = \chi(x_n < G(t) \le x_{n+1})$ are sluggish, Corollary 4.1 implies

$$\varliminf_{t\to\infty} \left[\frac{\tilde{E} e^{-\sigma\zeta_t} G(t)\chi(G(t) \in B_\Delta)}{\tilde{E} e^{-\sigma\zeta_t}} - \sum_{n=0}^{N-1} x_n \tilde{P}\{x_n < G(t) \le x_{n+1}\}\right] \ge 0$$

and

$$\varliminf_{t\to\infty} \left[\sum_{n=0}^{N-1} x_{n+1} \tilde{P}\{x_n < G(t) \le x_{n+1}\} - \frac{\tilde{E} e^{-\sigma\zeta_t} G(t)\chi(G(t) \in B_\Delta)}{\tilde{E} e^{-\sigma\zeta_t}}\right] \ge 0$$

$$(4.2.22)$$

Letting the mesh m approach 0, we find

$$\lim_{t\to\infty} \left[\frac{\tilde{E} e^{-\sigma\zeta_t} G(t)\chi(G(t) \in B_\Delta)}{\tilde{E} e^{-\sigma\zeta_t}} - \tilde{E} G(t)\chi(G(t) \in B_\Delta)\right] = 0. \qquad (4.2.23)$$

(4.2.20) and (4.2.23) guarantee the result. □

The following lemma is useful for checking that the conditions of Lemma 4.5 are fulfilled.

Lemma 4.6

Let G(t) be a natural process having proper distribution J_t. If $J_t \xrightarrow{w} J$ and

$$\int |x| dJ_t(x) \to \int |x| dJ(x) < \infty \tag{4.2.24}$$

then for every $\varepsilon > 0$ there is a $\Delta(\varepsilon)$ making $\int_{\{|x|>\Delta\}} |x| dJ_t(x) < \varepsilon$ for all sufficiently large t.

Proof:

Fix $\varepsilon > 0$. There is a T_1 such that

$$\left| \int |x| dJ_t(x) - \int |x| dJ(x) \right| < \varepsilon \quad \text{for } t \geq T_1 \tag{4.2.25}$$

by (4.2.24).

There is a $\Delta(\varepsilon)$ such that $\int_{\{|x|>\Delta\}} |x| dJ(x) < \varepsilon$, (4.2.26)

where $\pm\Delta$ are continuity points of J. Define $g(x) = |x|$, $|x| \leq \Delta$, Δ otherwise. g is a bounded continuous function and thus

$$\int g(x) dJ_t(x) \to \int g(x) dJ(x).$$

Thus for $t \geq T_2$,

$$\left| \int g(x) dJ_t(x) - \int g(x) dJ(x) \right| < \varepsilon. \tag{4.2.27}$$

Because $\pm\Delta$ are fixed continuity points of J,

$$1-J_t(\Delta) + J_t(-\Delta) \to 1-J(\Delta) + J(-\Delta).$$

In particular, for $t \geq T_3$,

$$\Delta \left| [1-J_t(\Delta)+J_t(-\Delta)] - [1-J(\Delta)+J(-\Delta)] \right| < \varepsilon. \tag{4.2.28}$$

Let $T_4 = \max(T_1, T_2, T_3)$. Then

$$\left| \int g(x)\,dJ_t(x) - \int g(x)\,dJ(x) \right| + \left| \int |x|\,dJ_t(x) - \int |x|\,dJ(x) \right| < 2\varepsilon$$

for $t \geq T_4$ by (4.2.27) and (4.2.25).

$$\Rightarrow \left| \int_{\{|x|>\Delta\}} [|x|-\Delta]\,dJ_t(x) - \int_{\{|x|>\Delta\}} [|x|-\Delta]\,dJ(x) \right| < 2\varepsilon \tag{4.2.29}$$

$$\Rightarrow \int_{\{|x|>\Delta\}} [|x|-\Delta]\,dJ_t(x) < 3\varepsilon \tag{4.2.30}$$

for $t \geq T_4$ by (4.2.26) and (4.2.30). But $\Delta \int_{\{|x|>\Delta\}} dJ_t(x) < 2\varepsilon$ by (4.2.28) and (4.2.26) and hence

$$\int_{\{|x|>\Delta\}} |x|\,dJ_t(x) < 5\varepsilon \quad \text{for all } t \geq T_4. \qquad \square$$

4.3 RESULTS AND EXAMPLES

We now use the technical lemmas of the last section to derive some results about transient cumulative processes. We also return to the "time until ruin" problem discussed in Chapter 1.

Lemma 4.7

Suppose $W(t)$ is a cumulative process such that $\tilde{\mu}_1 < \infty$ and $\tilde{\kappa}_2^* < \infty$. Then

$$\eta_1(t) \equiv \frac{W(t)-\tilde{\kappa}_1 N(t)}{\tilde{\sigma}_y\sqrt{\frac{t}{\tilde{\mu}_1}}} \tag{4.3.1}$$

is a sluggish process.

Proof:

From (1.2.4), we know that $\tilde{P}\{\eta_1(t)\le\alpha\} \to \Phi(\alpha)$ and hence by Lemma 4.4 it is sufficient to prove that $\eta_1(t+T) - \eta_1(t) \overset{\tilde{P}}{\to} 0$.

$$\tilde{P}\left\{\left| \frac{W(t+T)-\tilde{\kappa}_1 N(t+T)}{\tilde{\sigma}_y\sqrt{\frac{t+T}{\tilde{\mu}_1}}} - \frac{W(t)-\tilde{\kappa}_1 N(t)}{\tilde{\sigma}_y\sqrt{\frac{t}{\tilde{\mu}_1}}} \right| > \varepsilon\right\}$$

$$= \tilde{P}\left\{\left| \frac{W(t+T)-W(t)}{\tilde{\sigma}_y\sqrt{\frac{t+T}{\tilde{\mu}_1}}} - \frac{\tilde{\kappa}_1[N(t+T)-N(t)]}{\tilde{\sigma}_y\sqrt{\frac{t+T}{\tilde{\mu}_1}}} - \right.\right.$$

$$\left.\left. \frac{W(t)-\tilde{\kappa}_1 N(t)}{\tilde{\sigma}_y\sqrt{\frac{t}{\tilde{\mu}_1}}}\left(1 - \sqrt{\frac{t}{t+T}}\right)\right| > \varepsilon\right\}. \tag{4.3.2}$$

Hence

$$\tilde{P}\{|\eta_1(t+T)-\eta_1(t)|> \varepsilon \}$$

$$\le \tilde{P}\left\{\left| \frac{W(t+T)-W(S_{N(t+T)})+W(S_{N(t+T)})-W(S_{N(t)+1})+W(S_{N(t)+1})-W(t)}{\tilde{\sigma}_y\sqrt{\frac{t+T}{\tilde{\mu}_1}}}\right| >\frac{\varepsilon}{3}\right\}$$

$$+ \tilde{P}\left\{\left|\frac{\tilde{\kappa}_1[N(t+T)-N(t)]}{\tilde{\sigma}_y\sqrt{\frac{t+T}{\tilde{\mu}_1}}}\right| > \frac{\varepsilon}{3}\right\} + \tilde{P}\left\{\left|\frac{W(t)-\tilde{\kappa}_1 N(t)}{\tilde{\sigma}_y\sqrt{\frac{t}{\tilde{\mu}_1}}}\right| > \frac{\varepsilon\sqrt{t+T}}{3(\sqrt{t+T}-\sqrt{t})}\right\}$$

$$= A + B + C. \tag{4.3.3}$$

In his paper introducing the concept of cumulative processes, Smith (1955) proved that if $\tilde{\mu}_1 < \infty$ and $\tilde{\kappa}_2^* < \infty$,

(i) $\dfrac{W(t)-W(S_{N(t)})}{\sqrt{t}} \to 0$ a.s. $(\tilde{P})$ and

(ii) $\dfrac{W(S_{N(t)+1})-W(t)}{\sqrt{t}} \to 0$ a.s. $(\tilde{P})$.

These two results coupled with the fact that

$$\frac{W(S_{N(t+T)})-W(S_{N(t)+1})}{\sqrt{t+T}} = \frac{\sum_{j=N(t)+2}^{N(t+T)} Y_j}{N(t+T)-N(t)-1} \cdot \frac{N(t+T)-N(t)-1}{\sqrt{t+T}} \to 0$$

a.s. $(\tilde{P})$ by the Strong Law of Large Numbers ensure that $A \to 0$. $B \to 0$ by the Strong Law also. Finally, $C \to 0$ by the asymptotic normality of $\eta_1(t)$. □

Lemma 4.8

Suppose $W(t)$ is a cumulative process such that $\tilde{\mu}_2 < \infty$ and $\tilde{\kappa}_2^* < \infty$. Then

$$\eta_2(t) = \frac{W(t) - \frac{\tilde{\kappa}_1 t}{\tilde{\mu}_1}}{\sqrt{\frac{\tilde{\gamma} t}{\tilde{\mu}_1}}} \quad \text{is sluggish.} \tag{4.3.4}$$

Proof:

(1.2.5) and obvious modifications to the argument in Lemma 4.7 will suffice. □

Theorem 4.1

Let W(t) be a cumulative process such that $\tilde{\mu}_1 < \infty$ and $\tilde{\kappa}_2^* < \infty$. Then

(i) $$P\left\{\frac{W(t)-\tilde{\kappa}_1 N(t)}{\tilde{\sigma}_y \sqrt{\frac{t}{\tilde{\mu}_1}}} \le \alpha \mid A(t)=1\right\} \to \Phi(\alpha) \tag{4.3.5}$$

(ii) If, addition, $\tilde{\mu}_2 < \infty$,

$$P\left\{\frac{W(t) - \frac{\tilde{\kappa}_1 t}{\tilde{\mu}_1}}{\sqrt{\frac{\tilde{\gamma} t}{\tilde{\mu}_1}}} \le \alpha \middle| A(t) = 1\right\} \to \Phi(\alpha). \tag{4.3.6}$$

Proof:

Using the notation $\eta_j(t)$ established in Lemmas 4.7 and 4.8, $\{\eta_j(t) \le \alpha\}$ is a sluggish event if condition j holds; j = 1, 2.

$$P\{\eta_j(t)\le\alpha \mid A(t)=1\} = \frac{\tilde{E}e^{-\sigma\zeta_t}\chi(\eta_j(t)\le\alpha)}{\tilde{E}e^{-\sigma\zeta_t}}$$

$$\rightarrow \Phi(\alpha) \text{ by Corollary 4.1.} \qquad \square$$

Let us re-examine the "time until ruin" problem. We found that

$$P\{\tau^*\le t\} = P\{\sum_{k=1}^{M(t)} L_k > u\}$$

where $M(t)$ is the renewal count for the transient renewal process $\{Z_i\}$. Note that $L_k^* \equiv L_k$. Let $J(z,\ell)$ be the joint distribution of Z_1 and L_1 and suppose there is a σ making

$$\int_0^\infty \int_0^\infty e^{\sigma z} dJ(z,\ell) = 1.$$

Define $d\tilde{J}(z,\ell) = e^{\sigma z}dJ(z,\ell)$. If $\tilde{E}Z_1{}^2 < \infty$ and $\tilde{E}L_1{}^2 < \infty$, then Theorem 4.1 implies

$$P\left\{\frac{\sum_{k=1}^{M(t)} L_k - \frac{\tilde{E}L_1 t}{\tilde{E}Z_1}}{\sqrt{\frac{\tilde{\gamma}t}{\tilde{E}Z_1}}} > \frac{u - \frac{\tilde{E}L_1 t}{\tilde{E}Z_1}}{\sqrt{\frac{\tilde{\gamma}t}{\tilde{E}Z_1}}} \;\middle|\; A(t) = 1\right\}$$

$$\rightarrow \Phi\left(\frac{\frac{\tilde{E}L_1 t}{\tilde{E}Z_1} - u}{\sqrt{\frac{\tilde{\gamma}t}{\tilde{E}Z_1}}}\right)$$

where $\tilde{\gamma} = \tilde{E}(L_1 - \frac{\tilde{E}L_1}{\tilde{E}Z_1} Z_1)^2$. Therefore

$$P\{\tau^* \le t\} \sim \frac{(1-\omega)e^{-\sigma t}}{\sigma \tilde{E}Z_1} \tag{4.3.7}$$

where

$$1-\omega = P\{Z_1 = \infty\}.$$

Equation (4.3.7) agrees with Cramer's estimate for the time until ruin (1955).

<u>Theorem 4.2</u>

Suppose W(t) is a cumulative process such that $\tilde{\mu}_2 < \infty$ and $\tilde{\kappa}_2^* < \infty$. Then

$$E[(W(t) - \frac{\tilde{\kappa}_1 t}{\tilde{\mu}_1})^2 | A(t) = 1] = \frac{\tilde{\gamma} t}{\tilde{\mu}_1} + o(t). \tag{4.3.8}$$

Proof:

Let

$$\eta_2(t) = \frac{W(t) - \frac{\tilde{\kappa}_1 t}{\tilde{\mu}_1}}{\sqrt{\frac{\tilde{\gamma} t}{\tilde{\mu}_1}}}, \text{ as before.}$$

$$\tilde{P}\{\eta_2(t)^2 \le \alpha\} \to 2\Phi(\sqrt{\alpha})-1, \ \alpha \ge 0 \text{ since}$$

$$\tilde{P}\{\eta_2(t) \le \alpha\} \to \Phi(\alpha).$$

Smith (1955) has shown that $\tilde{E}\eta_2(t)^2 \to 1$ which is, of course, the first moment of the limiting distribution of $\eta_2(t)^2$. Note that as $\eta_2(t)$ is a sluggish process, $\eta_2(t)^2$ is too. Therefore, by Lemma 4.5 and 4.6,

$$\frac{\tilde{E}e^{-\sigma\zeta_t}\eta_2(t)^2}{\tilde{E}e^{-\sigma\zeta_t}} - \tilde{E}\eta_2(t)^2 \to 0 \text{ as } t \to \infty. \tag{4.3.9}$$

Hence

$$E[\eta_2(t)^2|A(t) = 1] = \frac{\tilde{E}e^{-\sigma\zeta_t}\eta_2(t)^2}{\tilde{E}e^{-\sigma\zeta_t}} \to 1. \qquad \square$$

In light of this result, it is reasonable to ask whether

$$\mathrm{Var}[W(t)|A(t) = 1] = \frac{\tilde{\gamma}t}{\tilde{\mu}_1} + o(t). \tag{4.3.10}$$

We have

$$E[\{W(t)-E(W(t)|A(t)=1) + E(W(t)|A(t)=1)- \frac{\tilde{\kappa}_1 t}{\tilde{\mu}_1}\}^2|A(t)=1]$$

$$= E[\{W(t)-E(W(t)|A(t)=1)\}^2|A(t)=1]$$

$$+ \{E(W(t)|A(t)=1) - \frac{\tilde{\kappa}_1 t}{\tilde{\mu}_1}\}^2 = \frac{\tilde{\gamma}t}{\tilde{\mu}_1} + o(t).$$

If we can show that

$$\{E(W(t)|A(t)=1) - \frac{\tilde{\kappa}_1 t}{\tilde{\mu}_1}\} = o(\sqrt{t}),$$

then we will know

$$\mathrm{Var}[W(t)|A(t)=1] = \frac{\tilde{\gamma}t}{\tilde{\mu}_1} + o(t).$$

In the next two chapters we will derive the asymptotic forms of the conditional cumulants of $W(t)$; in the process we will discover conditions under which (4.3.10) does indeed hold.

CHAPTER 5

THE CUMULANTS OF TRANSIENT RENEWAL PROCESSES

5.1 PRELIMINARY RESULTS

We shall show that when the proper distribution $\tilde{F}$ exists, the kth conditional cumulant of $N(t)$ is asymptotically linear in t, just as is the kth cumulant of the renewal count of a proper process. To deal with cumulants, Smith (1959) had to assume that (the then proper) $F \in \mathcal{C}$, the class of distributions for which $F^{(n)}$ has an absolutely continuous component for some finite n. We will need $\tilde{F} \in \mathcal{C}$ and will see that it is sufficient that $F(\infty)^{-1}F \in \mathcal{C}$.

Lemma 5.1

Suppose $d\tilde{F}(x) = e^{\sigma x}dF(x)$. Then

$$\tilde{F}^{(n)}(x) = \int_0^x e^{\sigma y} dF^{(n)}(y). \tag{5.1.1}$$

Proof:

$$\tilde{F}^*(s) = F^*(s-\sigma) \Rightarrow \tilde{F}^*(s)^n = F^*(s-\sigma)^n.$$

That is,

$$\int_0^\infty e^{-sx} d\tilde{F}^{(n)}(x) = \int_0^\infty e^{-sx} e^{\sigma x} dF^{(n)}(x). \tag{5.1.2}$$

By the uniqueness theorem for Laplace Transforms,

$$\tilde{F}^{(n)}(x) = \int_0^x e^{\sigma y} dF^{(n)}(y). \qquad \square$$

<u>Lemma 5.2</u>

If $F(\infty)^{-1}F \in C$, $\tilde{F} \in C$.

Proof:

This is clear from Lemma 5.1, for if $F^{(n)}(x)$ has an absolutely continuous component $\int_0^x f_n(y)dy$, $\tilde{F}^{(n)}(x)$ has an absolutely continuous component $\int_0^x e^{\sigma y} f_n(y)dy$. $\square$

As a first step toward an examination of the conditional cumulants, let us consider

$$E[N(t)^k | A(t)=1] = \frac{\tilde{E}e^{-\sigma\zeta_t} N(t)^k}{\tilde{E}e^{-\sigma\zeta_t}} \qquad k = 1, 2, \ldots$$

<u>Lemma 5.3</u>

$$\tilde{E}e^{-\sigma\zeta_t} N(t)^k = \tilde{E}e^{-\sigma\zeta_t} {}_{*}EN(t)^k.$$

Proof:

$$\tilde{E}e^{-\sigma\zeta_t} N(t)^k = \sum_{n=1}^{\infty} n^k \int_0^t \int_{t-\tau}^{\infty} e^{-\sigma(y+\tau-t)} d\tilde{F}(y) d\tilde{F}^{(n)}(\tau)$$

$$= \sum_{n=1}^{\infty} n^k \int_0^t \int_{t-\tau}^{\infty} e^{\sigma t} dF(y) dF^{(n)}(\tau) \qquad \text{by Lemma 5.1}$$

$$= e^{\sigma t} \sum_{n=1}^{\infty} n^k [\omega F^{(n)}(t) - F^{(n+1)}(t)]. \qquad (5.1.3)$$

$$\tilde{E}e^{-\sigma\zeta_t} = \int_t^\infty e^{-\sigma(\tau-t)}d\tilde{F}(\tau) + \int_0^t\int_{t-\tau}^\infty e^{-\sigma(y+\tau-t)}d\tilde{F}(y)d\tilde{H}(\tau)$$

$$= e^{\sigma t}\int_t^\infty dF(\tau) + e^{\sigma t}\int_0^t\int_{t-\tau}^\infty dF(y)dH(\tau)$$

$$= e^{\sigma t}[\omega-(1-\omega)H(t)]. \tag{5.1.4}$$

$$\tilde{E}N(t)^k = \sum_{n=1}^\infty n^k[\tilde{F}^{(n)}(t)-\tilde{F}^{(n+1)}(t)]. \tag{5.1.5}$$

Hence

$$\tilde{E}e^{-\sigma\zeta_t} * \tilde{E}N(t)^k =$$

$$\int_0^t e^{\sigma(t-\tau)}[\omega-(1-\omega)H(t-\tau)]\sum_{n=1}^\infty n^k[d\tilde{F}^{(n)}(\tau)-d\tilde{F}^{(n+1)}(\tau)]$$

$$= e^{\sigma t}\sum_{n=1}^\infty n^k\int_0^t[\omega-(1-\omega)H(t-\tau)][dF^{(n)}(\tau)-dF^{(n+1)}(\tau)]$$

$$= e^{\sigma t}\sum_{n=1}^\infty n^k[\omega F^{(n)}(t)-\omega F^{(n+1)}(t)-(1-\omega)F^{(n+1)}(t)]$$

$$= e^{\sigma t}\sum_{n=1}^\infty n^k[\omega F^{(n)}(t)-F^{(n+1)}(t)]. \tag{5.1.6}$$

□

Define

$$M_k(t) = \tilde{E}N(t)_{(k)} = \tilde{E}N(t)[N(t)-1]\cdots[N(t)-k+1] \tag{5.1.7}$$

<u>Lemma 5.4</u>

$$M_k(t) = k!\tilde{H}^{(k)}(t).$$

Proof:

$$M_k(t) = \sum_{j=k}^{\infty} j_{(k)} \int_0^t \int_{t-\tau}^{\infty} d\tilde{F}(y)\, d\tilde{F}^{(j)}(\tau) \tag{5.1.8}$$

$$M_k{}^0(s) = \sum_{j=k}^{\infty} j_{(k)} \int_0^{\infty} e^{-st} \int_0^t \int_{t-\tau}^{\infty} d\tilde{F}(y)\, d\tilde{F}^{(j)}(\tau)\, dt$$

$$= \sum_{j=k}^{\infty} j_{(k)} \int_0^{\infty} \int_0^{\infty} \int_{\tau}^{y+\tau} e^{-st}\, dt\, d\tilde{F}(y)\, d\tilde{F}^{(j)}(\tau).$$

$$= \frac{1}{s} \sum_{j=k}^{\infty} j_{(k)} \int_0^{\infty} e^{-s\tau} \int_0^{\infty} (1-e^{-sy})\, d\tilde{F}(y)\, d\tilde{F}^{(j)}(\tau)$$

$$= s^{-1}(1-\tilde{F}^*(s)) \sum_{j=k}^{\infty} j_{(k)} \tilde{F}^*(s)^j = \frac{k!\tilde{F}^*(s)^k}{s[1-\tilde{F}^*(s)]^k}$$

$$= s^{-1} k! \tilde{H}^*(s)^k. \tag{5.1.9}$$

Since $M_k(t)$ is nondecreasing, its Laplace-Stieltjes Transform exists and is

$$M_k{}^*(s) = sM_k{}^0(s) = k!\tilde{H}^*(s)^k. \tag{5.1.10}$$

The uniqueness theorem for Laplace Transforms implies the result.

□

Combining Lemmas 5.3 and 5.4, we discover

$$E[N(t)_{(k)} \mid A(t)=1] = k!\int_0^t \frac{\tilde{E}e^{-\sigma\zeta_{t-\tau}}\, d\tilde{H}^{(k)}(\tau)}{\tilde{E}e^{-\sigma\zeta_t}}. \tag{5.1.11}$$

We know about the properties of $\tilde{H}^{(k)}$ from Smith (1959), but we need to find the rate at which $\tilde{E}e^{-\sigma\zeta_t}$ converges to $\tilde{K}^*(\sigma)$. The next lemma provides valuable information on this point.

<u>Lemma 5.5</u>

Suppose $m \geq 1$, $\tilde{\mu}_{m+1} < \infty$, and $\tilde{F} \in \mathcal{C}$. Then

$$\tilde{E}e^{-\sigma\zeta_t} - \tilde{K}^*(\sigma) = o(t^{-m}).$$

Proof:

$$\tilde{E}e^{-\sigma\zeta_t} = \int_t^\infty e^{-\sigma(x-t)} d\tilde{F}(x) + \int_0^t \int_{t-\tau}^\infty e^{-\sigma(x+\tau-t)} d\tilde{F}(x) d\tilde{H}(\tau)$$

$$= o(t^{-m-1}) + \int_0^t \int_{t-x}^t e^{-\sigma(x+\tau-t)} d\tilde{H}(\tau) d\tilde{F}(x) + \int_t^\infty \int_0^t e^{-\sigma(x+\tau-t)} d\tilde{H}(\tau) d\tilde{F}(x). \tag{5.1.12}$$

Note that

$$\int_t^\infty \int_0^t e^{-\sigma(x+\tau-t)} d\tilde{H}(\tau) d\tilde{F}(x)$$

$$\leq \int_t^\infty e^{-\sigma(x-t)} \sum_{j=0}^{[t]} e^{-\sigma j} [\tilde{H}(j+1) - \tilde{H}(j)] d\tilde{F}(x)$$

$$\leq \kappa(1-e^{-\sigma})^{-1} \int_t^\infty e^{-\sigma(x-t)} d\tilde{F}(x) = o(t^{-m-1}).$$

κ is a finite upper bound on $\tilde{H}(j+1) - \tilde{H}(j)$; this bound [illegible] well-known inequality.

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Hence

$$\tilde{E}e^{-\sigma\zeta_t} = \int_0^t \int_0^x e^{-\sigma u}\tilde{H}(du+t-x)d\tilde{F}(x) + o(t^{-m-1}). \qquad (5.1.14)$$

$$\tilde{K}^*(\sigma) = \frac{1}{\tilde{\mu}_1}\int_0^\infty \int_x^\infty e^{-\sigma(y-x)}d\tilde{F}(y)dx$$

$$= \frac{1}{\tilde{\mu}_1}\int_0^t \int_0^y e^{-\sigma(y-x)}dx\ d\tilde{F}(y) + \frac{1}{\tilde{\mu}_1}\int_t^\infty \int_0^y e^{-\sigma(y-x)}dx\ d\tilde{F}(y)$$

$$= \frac{1}{\tilde{\mu}_1}\int_0^t \int_0^y e^{-\sigma u}du\ d\tilde{F}(y) + o(t^{-m-1}). \qquad (5.1.15)$$

Therefore

$$|\tilde{E}e^{-\sigma\zeta_t} - \tilde{K}^*(\sigma)| =$$

$$\left|\int_0^t \int_0^x e^{-\sigma u}[\tilde{H}(du+t-x) - \frac{du}{\tilde{\mu}_1}]d\tilde{F}(x)\right| + o(t^{-m-1})$$

$$\le \int_0^t \sum_{j=0}^{[x]} e^{-\sigma j}|\tilde{H}(t+j+1-x)-\tilde{H}(t+j-x) - \frac{1}{\tilde{\mu}_1}|d\tilde{F}(x) + o(t^{-m-1})$$

$$\le \int_0^{t/2} \sum_{j=0}^{[x]} e^{-\sigma j}|\tilde{H}(t+j+1-x)-\tilde{H}(t+j-x) - \frac{1}{\tilde{\mu}_1}|\ d\tilde{F}(x)$$

$$+ \kappa(1-e^{-\sigma})^{-1}\int_{t/2}^t d\tilde{F}(x) + o(t^{-m-1}). \qquad (5.1.16)$$

On the range $0 \le x \le \frac{t}{2}$, $|\tilde{H}(t+j+1-x)-\tilde{H}(t+j-x) - \frac{1}{\tilde{\mu}_1}|$ is $o(t^{-m})$ by Theorems 3 and 4 of Smith (1959) since $\tilde{\mu}_{m+1} < \infty$. Thus the first summand in (5.1.16) is $o(t^{-m})$ and the second is $o(t^{-m-1})$. The result follows. □

In Lemma 5.2 we found that $\tilde{E}e^{-\sigma\zeta_t} = e^{\sigma t}[\omega-(1-\omega)H(t)]$. This function is of bounded variation in every finite interval and thus its Laplace-Stieltjes Transform exists.

Lemma 5.6

$$L\{\tilde{E}e^{-\sigma\zeta_t}\} = \frac{s[\tilde{F}^*(s)-\omega]}{(\sigma-s)[1-\tilde{F}^*(s)]}\ .$$

Proof:

$$s^{-1}L\{\tilde{E}e^{-\sigma\zeta_t}\} = L^0\{\tilde{E}e^{-\sigma\zeta_t}\}$$

$$= \int_0^\infty \int_t^\infty e^{-st}e^{-\sigma(x-t)}d\tilde{F}(x)dt + \int_0^\infty \int_0^t \int_{t-\tau}^\infty e^{-st}e^{-\sigma(y+\tau-t)}d\tilde{F}(y)d\tilde{H}(\tau)dt$$

$$= \int_0^\infty e^{-\sigma x} \int_0^x e^{t(\sigma-s)}dt\ d\tilde{F}(x) + \int_0^\infty e^{-\sigma\tau}\int_0^\infty e^{-\sigma y} \int_\tau^{\tau+y} e^{t(\sigma-s)}dt\ d\tilde{F}(y)d\tilde{H}(\tau)$$

$$= (\sigma-s)^{-1} \int_0^\infty (e^{-sx}-e^{-\sigma x})d\tilde{F}(x) + (\sigma-s)^{-1}\int_0^\infty e^{-s\tau}\int_0^\infty (e^{-sy}-e^{-\sigma y})d\tilde{F}(y)d\tilde{H}(\tau)$$

$$= (\sigma-s)^{-1}[\tilde{F}^*(s)-\omega][1 + \tilde{H}^*(s)]$$

$$= \frac{F^*(s)-\omega}{(\sigma-s)[1-\tilde{F}^*(s)]} \quad \text{since } \tilde{H}^*(s) = \frac{\tilde{F}^*(s)}{1-\tilde{F}^*(s)}\ . \tag{5.1.17}$$

□

Returning to expression (5.1.11), we have

$$E[N(t)_{(k)}|A(t)=1] = k!\tilde{H}^{(k)}(t)+k!\int_0^t [\frac{\tilde{E}e^{-\sigma\zeta_{t-\tau}}-\tilde{E}e^{-\sigma\zeta_t}}{\tilde{E}e^{-\sigma\zeta_t}}]d\tilde{H}^{(k)}(\tau). \tag{5.1.18}$$

From Smith (1959), $\tilde{H}^{(k)}(t)$ is asymptotically a k<u>th</u> degree polynomial in t. Given Lemma 5.5, it is reasonable to expect $E[N(t)_k|A(t)=1]$ (and hence $E[N(t)^k|A(t)=1]$) to be a k<u>th</u> degree polynomial in t whose leading coefficient is the leading coefficient in $k!\tilde{H}^{(k)}(t)$ if $\tilde{\mu}_{k+1} < \infty$. Our arguments to this effect depend upon the following lemma.

<u>Lemma D</u> (Smith 1959)

A necessary and sufficient condition for the proper distribution F of a non-negative random variable to have its first k moments finite is that there exist another distribution function $F_{(k)}$ such that

$$F^*(s) = 1-\mu_1 s + \frac{\mu_2}{2!}s^2 - \cdots + \frac{\mu_{k-1}}{(k-1)!}(-s)^{k-1} + \frac{\mu_k}{k!}(-s)^k F^*_{(k)}(s)$$

for real $s > 0$. (5.1.19)

In fact, $F^*(s) = 1-\mu_1 s + \cdots + \frac{\mu_r}{r!}(-s)^r F^*_{(r)}(s)$ for $1 \le r \le k$. $F_{(r)}$, the r<u>th</u> derived distribution of F, has k-r finite moments. The j<u>th</u> moment of $F_{(r)}$ is a rational function of $\mu_1, \ldots, \mu_{r+j}$.

<u>Theorem 5.1</u>

Suppose $\tilde{F} \in C$ and $\tilde{\mu}_2 < \infty$. Then

$$E[N(t)|A(t)=1] = \frac{t}{\tilde{\mu}_1} + \frac{\tilde{\mu}_2}{\tilde{\mu}_1^2} + \frac{1}{\sigma\tilde{\mu}_1} - \frac{2-\omega}{1-\omega} + o(1).$$

Proof:

From (5.1.11) we know that

$$E[N(t)|A(t)=1] = \int_0^t \frac{\tilde{E}e^{-\sigma\zeta_{t-\tau}}d\tilde{H}(\tau)}{\tilde{E}e^{-\sigma\zeta_t}} . \tag{5.1.20}$$

Consider

$$\psi(t) = \int_0^t \tilde{E}e^{-\sigma\zeta_{t-\tau}}d\tilde{H}(\tau) - \tilde{K}^*(\sigma)\tilde{H}(t). \tag{5.1.21}$$

$$\psi^*(s) = \frac{s[\tilde{F}^*(s)-\omega]\tilde{H}^*(s)}{(\sigma-s)[1-\tilde{F}^*(s)]} - \tilde{K}^*(\sigma)\tilde{H}^*(s) \text{ by Lemma 5.6}$$

$$= \tilde{H}^*(s)\left[\frac{s[\tilde{F}^*(s)-\omega]}{(\sigma-s)[1-\tilde{F}^*(s)]} - \frac{1-\omega}{\sigma\tilde{\mu}_1}\right]. \tag{5.1.22}$$

By Lemma D,

$$\tilde{F}^*(s) = 1-\tilde{\mu}_1 s\tilde{F}^*_{(1)}(s) = 1-\tilde{\mu}_1 s + \frac{\tilde{\mu}_2}{2} s^2\tilde{F}^*_{(2)}(s),$$

which implies

$$\psi^*(s) = \tilde{H}^*(s)\left[\frac{s(1-\omega)}{(\sigma-s)\tilde{\mu}_1}\right]\left[\frac{\tilde{\mu}_2\tilde{F}^*_{(2)}(s)}{2\tilde{\mu}_1\tilde{F}^*_{(1)}(s)} + \frac{1}{\sigma} - \frac{\tilde{\mu}_1}{1-\omega}\right]. \tag{5.1.23}$$

$$\lim_{s\downarrow 0} \psi^*(s) = \frac{1-\omega}{\sigma\tilde{\mu}_1}\left[\frac{\tilde{\mu}_2}{2\tilde{\mu}_1} + \frac{1}{\sigma} - \frac{\tilde{\mu}_1}{1-\omega}\right] \lim_{s\downarrow 0} s\tilde{H}^*(s).$$

But

$$\lim_{s\downarrow 0} s\tilde{H}^*(s) = \lim_{s\downarrow 0} \frac{s\tilde{F}^*(s)}{1-\tilde{F}^*(s)} = \lim_{s\downarrow 0} \frac{\tilde{F}^*(s)}{\tilde{\mu}_1\tilde{F}^*_{(1)}(s)} = \tilde{\mu}_1^{-1}.$$

Therefore

$$\lim_{t\to\infty} \psi(t) = \frac{1-\omega}{\sigma\tilde{\mu}_1}\left[\frac{\tilde{\mu}_2}{2\tilde{\mu}_1^2} + \frac{1}{\sigma\tilde{\mu}_1} - \frac{1}{1-\omega}\right] \tag{5.1.24}$$

and hence

$$E[N(t)|A(t)=1] = \frac{\tilde{K}^*(\sigma)\{\tilde{H}(t) + \frac{\tilde{\mu}_2}{2\tilde{\mu}_1{}^2} + \frac{1}{\sigma\tilde{\mu}_1} - \frac{1}{1-\omega}\}}{\tilde{E}e^{-\sigma\zeta_t}} + o(1). \tag{5.1.25}$$

From Smith (1954),

$$\tilde{H}(t) = \frac{t}{\tilde{\mu}_1} + \frac{\tilde{\mu}_2}{2\tilde{\mu}_1^2} - 1 + o(1)$$

and Lemma 5.5 guarantees $t[\tilde{E}e^{-\sigma\zeta_t} - \tilde{K}^*(\sigma)] \to 0$. Thus

$$E[N(t)\ A(t)=1] = \frac{t}{\tilde{\mu}_1} + \frac{\tilde{\mu}_2}{\tilde{\mu}_1^2} + \frac{1}{\sigma\tilde{\mu}_1} - \frac{2-\omega}{1-\omega} + o(1). \qquad \square$$

The method employed in Theorem 5.1 can be used to derive forms for higher cumulants. However, the computations involved become increasingly burdensome as the order increases. For example, to find the conditional variance, we begin with

$$E[N(t)(N(t)-1)|A(t)=1] = 2\int_0^t \frac{\tilde{E}e^{-\sigma\zeta_{t-\tau}}d\tilde{H}^{(2)}(\tau)}{\tilde{E}e^{-\sigma\zeta_t}}. \tag{5.1.26}$$

If $\tilde{\mu}_3 < \infty$, we can show that

$$\int_0^t \frac{\tilde{E}e^{-\sigma\zeta_{t-\tau}}d\tilde{H}^{(2)}(\tau)}{\tilde{E}e^{-\sigma\zeta_t}} - \tilde{H}^{(2)}(t) - \tilde{H}(t)\left[\frac{\tilde{\mu}_2}{2\tilde{\mu}_1^2} + \frac{1}{\sigma\tilde{\mu}_1} - \frac{1}{1-\omega}\right]$$

$$\to \left(\frac{\tilde{\mu}_2}{2\tilde{\mu}_1^2} + \frac{1}{\sigma\tilde{\mu}_1} - \frac{1}{1-\omega}\right)\left(\frac{\tilde{\mu}_2}{2\tilde{\mu}_1^2} + \frac{1}{\sigma\tilde{\mu}_1} - 1\right) + \frac{\tilde{\mu}_2^2}{4\tilde{\mu}_1^2} - \frac{\tilde{\mu}_3}{6\tilde{\mu}_1^3}. \tag{5.1.27}$$

The argument depends heavily upon Lemma D and the distributions $\tilde{F}_{(1)}$, $\tilde{F}_{(2)}$, and $\tilde{F}_{(3)}$. Combining this result with Theorem 5.1 and Smith's work on the cumulants of proper renewal processes, we discover

$$E[N(t)^2 - (E[N(t)|A(t)=1]^2|A(t)=1] =$$

$$\left(\frac{\tilde{\mu}_2-\tilde{\mu}_1^2}{\tilde{\mu}_1^3}\right)t + \frac{2\tilde{\mu}_2^2}{\tilde{\mu}_1^4} - \frac{\tilde{\mu}_3}{\tilde{\mu}_1^3} - \frac{\tilde{\mu}_2}{\tilde{\mu}_1^2} + \frac{\tilde{\mu}_2}{\sigma\tilde{\mu}_1^3} - \frac{1}{\sigma\tilde{\mu}_1} + \frac{1}{\sigma^2\tilde{\mu}_1^2} - \frac{\omega}{(1-\omega)^2} + o(1). \tag{5.1.28}$$

In the next sections we provide a preferable method of deriving the cumulants. We prove they are asymptotically linear in t and find properties of their error terms.

5.2 THE CONDITIONAL ϕ-MOMENTS

Smith discovered asymptotic formulas for the cumulants of a renewal process by studying what he called the ϕ-moments of the process:

$$\phi_r(t) = E[N(t)+1][N(t)+2]\cdots[N(t)+r]. \tag{5.2.1}$$

These moments have generating function

$$\Phi_t(\zeta) = E[\frac{1}{(1-\zeta)^{N(t)+1}}] .$$

If $\tilde{\mu}_n < \infty$, then

$$\Phi_t(\zeta) = 1 + \sum_{j=1}^{n} \phi_j(t) \frac{\zeta^j}{j!} + o(\zeta^n) \text{ as } \zeta \to 0 \tag{5.2.2}$$

and $\Psi_t(\zeta) = \log \Phi_t(\zeta)$ admits the expansion

$$\sum_{j=1}^{n} \psi_j(t) \frac{\zeta^j}{j!} + o(\zeta^n). \tag{5.2.3}$$

The coefficients $\psi_j(t)$ are called ψ-cumulants of $N(t)$. By working with ϕ-moments and ψ-cumulants rather than conventional moments and cumulants, one bypasses certain computational complexities. The rth conventional cumulant $k_r(t)$ is a linear combination of $\psi_1(t),\cdots,\psi_r(t)$.

$$k_1(t) = \psi_1(t)-1$$

$$k_n(t) = \sum_{j=0}^{n-1} (-1)^j t_{n,n-j}\psi_{n-j}(t), \; n \geq 2 \tag{5.2.4}$$

where the t_n's are Stirling's numbers of the second kind. See Smith (1959) for details.

We will study the conditional ϕ-moments and ψ-cumulants of transient renewal processes and then relate our results to the ordinary conditional cumulants, which we specify up to an error term. Let

$$\phi_{nc}(t) = E[(N(t)+1)\cdots(N(t)+n)\,|\,A(t)=1]$$

$$= \frac{\tilde{E}e^{-\sigma\zeta_t}[N(t)+1]\cdots[N(t)+n]}{\tilde{E}e^{-\sigma\zeta_t}} = \frac{\tilde{E}e^{-\sigma\zeta_t}*\tilde{\phi}_n(t)}{\tilde{E}e^{-\sigma\zeta_t}} \tag{5.2.5}$$

by Lemma 5.3.

Consider

$$L\{\tilde{E}e^{-\sigma\zeta_t} * \tilde{\phi}_n(t)\} = \frac{s[\omega-\tilde{F}^*(s)]\tilde{\phi}_n^*(s)}{(s-\sigma)[1-\tilde{F}^*(s)]} \tag{5.2.6}$$

by Lemma 5.6. Our goal is to rewrite (5.2.6) as a linear combination of $\tilde{\phi}_n^*(s)$, $\tilde{\phi}_{n-1}^*(s),\cdots,$ $\tilde{\phi}_1^*(s)$ because Smith carefully studied the properties of $\tilde{\phi}_n(t),\cdots,$ $\tilde{\phi}_1(t)$ in his 1959 paper.

Suppose n is a positive integer, p is nonnegative, and $\tilde{\mu}_{n+p+1} < \infty$. Assume temporarily that $s > \sigma$. Note that

$$\frac{\omega-\tilde{F}^*(s)}{s-\sigma} = \frac{\omega-\int_0^\infty e^{-(s-\sigma)x}dF(x)}{s-\sigma}$$

$$=\omega\int_0^\infty e^{-(s-\sigma)x}dx - (s-\sigma)^{-1}F^*(s-\sigma)$$

$$= \int_0^\infty e^{-sx}\{e^{\sigma x}[\omega-F(x)]\}dx \equiv \int_0^\infty e^{-sx}L(x)dx. \tag{5.2.7}$$

Since

$$\int_0^\infty e^{\sigma x}dF(x) = 1,\ L(\infty) = \lim_{x\to\infty} e^{\sigma x}[\omega-F(x)] = 0$$

and hence

$$\int_0^\infty x^{n+p+1}|dL(x)| = \int_0^\infty \int_0^x (n+p+1)y^{n+p}dy|dL(x)|$$

$$= (n+p+1)\int_0^\infty y^{n+p}L(y)dy = (n+p+1)\int_0^\infty y^{n+p}e^{\sigma y}[\omega-F(y)]dy.$$

$$= (n+p+1)\int_0^\infty \int_y^\infty y^{n+p} e^{\sigma y} dF(x)dy$$

$$< \int_0^\infty x^{n+p+1} e^{\sigma x} dF(x) = \tilde{\mu}_{n+p+1} < \infty. \tag{5.2.8}$$

Therefore

$$\frac{s[\omega-\tilde{F}^*(s)]}{s-\sigma} = sL^o(s) = L^*(s)$$

is the Laplace-Stieltjes Transforms of a function of bounded variation on $[0,\infty)$ having n+p+1 absolute moments. The transform $L^*(s)$ is defined throughout the half plane $R(s) > 0$; the integrability of $L(x)$ and the identity $L^*(s) = sL^o(s)$ imply that $L^*(s) = 0(|s|)$ as $|s| \to 0$. Thus given $\tilde{\mu}_{n+p+1} < \infty$ and $\tilde{F} \in C$, Theorem G of the appendix applies with $m = n + p$ and $q = 1$. It follows that

$$\frac{s[\omega-\tilde{F}^*(s)]}{(s-\sigma)[1-\tilde{F}^*(s)]} = \frac{L^*(s)}{1-\tilde{F}^*(s)} = \alpha_1 D_1^*(s) - \alpha_2 D_2^*(s),\ s > \sigma, \tag{5.2.9}$$

where D_1 and D_2 are distributions on $[0,\infty)$ having n+p finite moments. Notice that the right hand side of (5.2.9) is an analytic function throughout the half plane $R(s) > 0$. Thus equation (5.2.9) must hold for $R(s) > 0$.

Let the first n moments of D_1 and D_2 be $\nu_{11}, \nu_{12}, \cdots, \nu_{1n}$ and $\nu_{21}, \nu_{22}, \cdots, \nu_{2n}$. We can write

$$D_j^*(s) = 1-\nu_{j1}s + \frac{\nu_{j2}s^2}{2!} - \ldots + \frac{\nu_{jn}(-s)^n}{n!} D^*_{j(n)}(s) \tag{5.2.10}$$

where $D_{j(n)}$ is the nth derived distribution of D_j, $j = 1,2$. $D_{j(n)}$ has p finite moments.

Define

$$Q(s) = 1-\tilde{F}^*(s) = \tilde{\mu}_1 s - \frac{\tilde{\mu}_2 s^2}{2!} + \cdots - \frac{\tilde{\mu}_n(-s)^n}{n!} - \frac{\tilde{\mu}_{n+1}(-s)^{n+1}}{(n+1)!}\tilde{F}^*_{(n+1)}(s). \tag{5.2.11}$$

We can rewrite $D_j^*(s)$ as

$$D_j^*(s) = 1 + \lambda_{j1}Q + \cdots + \lambda_{jn}Q^n + R_{jn}^*(s) \tag{5.2.12}$$

where the λ_j's are chosen to make the coefficient of s^k in $\lambda_{j1}Q + \cdots + \lambda_{jn}Q^n$ equal to

$$\frac{\nu_{jk}(-1)^k}{k!},$$

the coefficient of s^k in $D_j^*(s)$, $1 \le k \le n$.

Let $L_j^*(s) = 1 + \lambda_{j1}Q + \cdots + \lambda_{jn}Q^n$, $j = 1, 2$.

Lemma 5.7

$L_j(t)$ is a linear combination of distribution functions on $[0,\infty)$ having $n + p + 1$ finite moments.

Proof:

$$L_j^*(s) = 1 + \sum_{k=1}^{n} \lambda_{jk}Q^k = 1 + \sum_{k=1}^{n} \lambda_{jk}[1-\tilde{F}^*(s)]^k$$

$$= 1 + \sum_{k=1}^{n} \lambda_{jk} \sum_{\ell=0}^{k} \binom{k}{\ell}(-1)^\ell \tilde{F}^{(\ell)*}(s)$$

$$= 1 + \sum_{k=1}^{n} \lambda_{jk} + \sum_{\ell=1}^{n} \tilde{F}^{(\ell)*}(s) \sum_{k=\ell}^{n} \binom{k}{\ell}(-1)^{\ell}\lambda_{jk}.$$

Define

$$\gamma_{j0} \equiv 1 + \sum_{k=1}^{n} \lambda_{jk};\ \gamma_{j\ell} = \sum_{k=\ell}^{n} \binom{k}{\ell}(-1)^{\ell}\lambda_{jk}.$$

Then

$$L_j^{*}(s) = \gamma_{j0} + \sum_{\ell=1}^{n} \gamma_{j\ell}\tilde{F}^{(\ell)*}(s)$$

and hence

$$L_j(t) = \gamma_{j0}U(t) + \sum_{\ell=1}^{n} \gamma_{j\ell}\tilde{F}^{(\ell)}(t). \tag{5.2.13}$$

We remark that

$$\int_0^{\infty} x^{n+p+1} d\tilde{F}^{(\ell)}(x) < \infty \text{ for } \ell \geq 1 \text{ since } \tilde{\mu}_{n+p+1} < \infty. \qquad \Box$$

As a result of the last lemma,

$$L_j^{*}(s) = 1+\alpha_{j1}s+\cdots+\alpha_{jn}s^{n}+\alpha_{j,n+1}s^{n+1}L_{j(n+1)}^{*}(s) \tag{5.2.14}$$

where $L_{j(n+1)}(t)$ is a linear combination of distributions having p finite moments. But from the definition of the λ_j's, we must have

$$\alpha_{jk} = \frac{\nu_{jk}(-1)^k}{k!}, \quad 1 \leq k \leq n.$$

Thus

$$L_j^*(s) = 1-\nu_{j1}s + \cdots + \frac{\nu_{jn}(-s)^n}{n!} + \alpha_{j,n+1}s^{n+1}L_{j(n+1)}^*(s). \quad (5.2.15)$$

From (5.2.10) and (5.2.15) we conclude

$$R_{jn}^*(s) = D_j^*(s)-L_j^*(s) = \frac{\nu_{jn}(-s)^n}{n!}[D_{j(n)}^*(s)-1]$$

$$- \alpha_{j,n+1}s^{n+1}L_{j(n+1)}^*(s). \quad (5.2.16)$$

Now define

$$\lambda_k \equiv \alpha_1\lambda_{1k} - \alpha_2\lambda_{2k}; \; \lambda_0 \equiv \alpha_1 - \alpha_2 \;;$$

$$R_n^*(s) \equiv \alpha_1 R_{1n}^*(s) - \alpha_2 R_{2n}^*(s).$$

By equations (5.2.9) and (5.2.12) we have

$$\frac{s[\omega-\tilde{F}^*(s)]}{(s-\sigma)[1-\tilde{F}^*(s)]} = \lambda_0 + \lambda_1 Q + \cdots + \lambda_n Q^n + R_n^*(s). \quad (5.2.17)$$

<u>Lemma E</u> (Smith 1959)

$$\tilde{\phi}_n^*(s) = n!Q(s)^{-n}.$$

<u>Lemma 5.8</u>

If $\tilde{\mu}_{n+p+1} < \infty$, $p \geq 0$, and $\tilde{F} \in C$, then

$$\frac{s[\omega-\tilde{F}^*(s)]\tilde{\phi}_n^*(s)}{(s-\sigma)[1-\tilde{F}^*(s)]} = \sum_{k=0}^{n} n_{(k)}\lambda_k\tilde{\phi}_{n-k}^*(s) + \frac{n!R_n^*(s)}{[1-\tilde{F}^*(s)]^n}.$$

Proof:

The expression follows directly from expression (5.2.17) and Lemma E. □

When we invert the transforms in Lemma 5.8, we will find $\tilde{E}e^{-\sigma\zeta_t} * \tilde{\phi}_n(t)$ is a linear combination of $\tilde{\phi}_n(t), \cdots, \tilde{\phi}_1(t)$, a constant term, plus an error term. We will prove the error term belongs to a particular class of functions.

Let $m \geq 0$ and $\Psi(t)$ be a function on $[0,\infty)$. We say $\Psi(t)$ belongs to the class $B(m)$ if and only if it is of bounded total variation and

$$\int_0^\infty t^m |d\Psi(t)| < \infty .$$

It is clear that if $\Psi(t) \in B(m)$ and $\Psi(\infty) = 0$, we may write

$$\Psi(t) = \frac{\lambda(t)}{(1+t)^m} \qquad (5.2.18)$$

where $\lambda(t)$ is of bounded variation, tends to zero as t approaches infinity, and if $m \geq 1$, $\frac{\lambda(t)}{1+t}$ belongs to the class L_1.

<u>Lemma 5.9</u>

If $\tilde{\mu}_{n+p+1} < \infty$, $p \geq 0$, and $\tilde{F} \in C$, then

$$\frac{R_n^*(s)}{[1-F^*(s)]^n} = \Psi^*(s)$$

where $\Psi(t) \in B(p)$ and $\Psi(\infty) = 0$.

Proof:

It is sufficient to show that $\frac{R_{jn}^*(s)}{[1-\tilde{F}^*(s)]^n}$ has the described

properties. Identify p with m and n with q in Theorem G. We have

1) $\tilde{F} \in C$ and $\tilde{\mu}_{p+1} < \infty$.

2) $R_{jn}(t) = D_j(t) - L_j(t)$ is a function of bounded variation on $[0,\infty)$ having $n + p$ absolute moments since D_j and L_j are such functions. Also,

$$R_{jn}^*(s) = o(|s|^n) \text{ as } |s| \to 0$$

by equation (5.2.16). Theorem G implies

$$\frac{R_{jn}^*(s)}{[1-\tilde{F}^*(s)]^n} = \Psi_j^*(s)$$

where $\Psi_j(t) \in B(p)$.

$$\Psi_j(\infty) = 0 \text{ since } \lim_{|s|\to 0} \frac{R_{jn}^*(s)}{[1-\tilde{F}^*(s)]^n} = 0. \qquad \Box$$

We now want to find λ_k explicitly for $k = 0,1,2,3$ and indicate its general form for higher values of k. We learned in equation (5.2.17) that when $\tilde{\mu}_{n+p+1} < \infty$.

$$\frac{s[\omega-\tilde{F}^*(s)]}{(s-\sigma)[1-\tilde{F}^*(s)]} = \lambda_0 + \lambda_1 Q + \cdots + \lambda_n Q^n + o(s^n) \text{ as } s \to 0.$$

First we rewrite

$$\frac{\omega-\tilde{F}^*(s)}{(s-\sigma)\tilde{\mu}_1 \tilde{F}^*_{(1)}(s)} = \sum_{j=0}^{n} c_j s^j + o(s^n) \text{ as } s \to 0. \tag{5.2.19}$$

We have derived c_j for $j \le 4$ by using properties of the derived distributions of $\tilde{F}$.

$$c_0 = \frac{1-\omega}{\sigma\tilde{\mu}_1}$$

$$c_1 = \frac{1-\omega}{\sigma^2\tilde{\mu}_1} - \frac{1}{\sigma} + \frac{(1-\omega)\tilde{\mu}_2}{2\sigma\tilde{\mu}_1^{\,2}}$$

$$c_2 = \frac{1-\omega}{\sigma^3\tilde{\mu}_1} - \frac{1}{\sigma^2} + \frac{(1-\omega)\tilde{\mu}_2}{2\sigma^2\tilde{\mu}_1^{\,2}} - \frac{(1-\omega)\tilde{\mu}_3}{6\sigma\tilde{\mu}_1^{2}} + \frac{(1-\omega)\tilde{\mu}_2^{\,2}}{4\sigma\tilde{\mu}_1^{\,3}}$$

$$c_3 = \frac{1-\omega}{\sigma^4\tilde{\mu}_1} - \frac{1}{\sigma^3} + \frac{(1-\omega)\tilde{\mu}_2}{2\sigma^3\tilde{\mu}_1^{\,2}} - \frac{(1-\omega)\tilde{\mu}_3}{6\sigma^2\tilde{\mu}_1^{\,2}} + \frac{(1-\omega)\tilde{\mu}_4}{24\sigma\tilde{\mu}_1^{\,2}} + \frac{(1-\omega)\tilde{\mu}_2^{\,2}}{4\sigma^2\tilde{\mu}_1^{\,3}}$$

$$- \frac{(1-\omega)\tilde{\mu}_2\tilde{\mu}_3}{6\sigma\tilde{\mu}_1^{\,3}} + \frac{(1-\omega)\tilde{\mu}_2^{\,3}}{8\sigma\tilde{\mu}_1^{\,4}}$$

$$c_4 = \frac{1-\omega}{\sigma^5\tilde{\mu}_1} - \frac{1}{\sigma^4} + \frac{(1-\omega)\tilde{\mu}_2}{2\sigma^4\tilde{\mu}_1^{\,2}} - \frac{(1-\omega)\tilde{\mu}_3}{6\sigma^3\tilde{\mu}_1^{\,2}} + \frac{(1-\omega)\tilde{\mu}_4}{24\sigma^2\tilde{\mu}_1^{\,2}} - \frac{(1-\omega)\tilde{\mu}_5}{120\sigma\tilde{\mu}_1^{\,2}}$$

$$+ \frac{(1-\omega)\tilde{\mu}_2^{\,2}}{4\sigma^3\tilde{\mu}_1^{\,3}} - \frac{(1-\omega)\tilde{\mu}_2\tilde{\mu}_3}{6\sigma^2\tilde{\mu}_1^{\,3}} + \frac{(1-\omega)\tilde{\mu}_2\tilde{\mu}_4}{24\sigma\tilde{\mu}_1^{\,3}} + \frac{(1-\omega)\tilde{\mu}_3^{\,2}}{36\sigma\tilde{\mu}_1^{\,3}} + \frac{(1-\omega)\tilde{\mu}_2^{\,3}}{8\sigma^2\tilde{\mu}_1^{\,4}}$$

$$- \frac{(1-\omega)\tilde{\mu}_2^{\,2}\tilde{\mu}_3}{8\sigma^2\tilde{\mu}_1^{\,4}} + \frac{(1-\omega)\tilde{\mu}_2^{\,4}}{16\sigma\tilde{\mu}_1^{\,5}} .$$

There are no conceptual difficulties in deriving c_5, c_6, ..., but the arithmetic involved becomes increasingly tedious.

Notice that

$$k!c_k = \frac{d^k}{ds^k}\left[\frac{\omega-\tilde{F}^*(s)}{(s-\sigma)\tilde{\mu}_1\tilde{F}_{(1)}{}^*(s)}\right]\Bigg|_{s=0}$$

and hence c_k is a rational function of the first k moments of $\tilde{F}$ and $\tilde{F}_{(1)}$. That is, $c_k = \tilde{\gamma}_{k+1}$, where $\tilde{\gamma}_{k+1}$ represents any rational function of $\tilde{\mu}_1, \tilde{\mu}_2, \cdots, \tilde{\mu}_{k+1}$.

Now to find the λ's, we equate coefficients of powers of s in the equation

$$\sum_{j=0}^{n} \lambda_j Q(s)^j + o(s^n) = \sum_{j=0}^{n} c_j s^j + o(s^n). \tag{5.2.20}$$

We find

$$\lambda_0 = \frac{1-\omega}{\sigma\tilde{\mu}_1}$$

$$\lambda_1 = \frac{1-\omega}{\sigma^2\tilde{\mu}_1{}^2} - \frac{1}{\sigma\tilde{\mu}_1} + \frac{(1-\omega)\tilde{\mu}_2}{2\sigma\tilde{\mu}_1{}^3}$$

$$\lambda_2 = \frac{1-\omega}{\sigma^3\tilde{\mu}_1{}^3} - \frac{1}{\sigma^2\tilde{\mu}_1{}^2} - \frac{\tilde{\mu}_2}{2\sigma\tilde{\mu}_1{}^3} + \frac{(1-\omega)\tilde{\mu}_2}{\sigma^2\tilde{\mu}_1{}^4} - \frac{(1-\omega)\tilde{\mu}_3}{6\sigma\tilde{\mu}_1{}^4} + \frac{(1-\omega)\tilde{\mu}_2{}^2}{2\sigma\tilde{\mu}_1{}^5}$$

$$\lambda_3 = \frac{1-\omega}{\sigma^2\tilde{\mu}_1{}^4} - \frac{1}{\sigma^3\tilde{\mu}_1{}^3} - \frac{\tilde{\mu}_2}{\sigma^2\tilde{\mu}_1{}^4} + \frac{3(1-\omega)\tilde{\mu}_2}{2\sigma^3\tilde{\mu}_1{}^5} + \frac{\tilde{\mu}_3}{6\sigma\tilde{\mu}_1{}^4} - \frac{(1-\omega)\tilde{\mu}_3}{3\sigma^2\tilde{\mu}_1{}^5}$$

$$+ \frac{(1-\omega)\tilde{\mu}_4}{24\sigma\tilde{\mu}_1{}^5} - \frac{\tilde{\mu}_2{}^2}{2\sigma\tilde{\mu}_1{}^5} + \frac{5(1-\omega)\tilde{\mu}_2{}^2}{4\sigma^2\tilde{\mu}_1{}^6} - \frac{5(1-\omega)\tilde{\mu}_2\tilde{\mu}_3}{12\sigma\tilde{\mu}_1{}^6} + \frac{5(1-\omega)\tilde{\mu}_2{}^3}{8\sigma\tilde{\mu}_1{}^7}.$$

In general, $\lambda_k = \tilde{\gamma}_{k+1}$ since $c_k = \tilde{\gamma}_{k+1}$ and the coefficient of s^k in

$$Q^k = [\tilde{\mu}_1 s\tilde{F}_{(1)}{}^*(s)]^k$$

will be of the form $\tilde{\gamma}_k$.

Lemma F (Smith (1959)

If $\tilde{\mu}_{n+p+1} < \infty$, $p \geq 0$ and $\tilde{F} \in \mathcal{C}$, then

$$\tilde{\phi}_n(t) = \tilde{\gamma}_1 t^n + \tilde{\gamma}_2 t^{n-1} + \cdots + \tilde{\gamma}_{n+1} + \frac{\lambda(t)}{(1+t)^p}$$

where $\lambda(t)$ is of bounded total variation, is $o(1)$ as $t \to \infty$, satisfies the condition

$$\lambda(t) - \lambda(t-\alpha) = o(t^{-1}) \text{ as } t \to \infty$$

for every fixed $\alpha > 0$, and when $p \geq 1$ has the additional property that $\frac{\lambda(t)}{1+t} \in L_1$.

Theorem 5.2

If $\tilde{\mu}_{n+p+1} < \infty$, $p \geq 0$, and $\tilde{F} \in \mathcal{C}$, then

$$E[(N(t)+1)(N(t)+2)\cdots(N(t)+n)\,|\,A(t)=1]$$

$$= \tilde{\gamma}_1 t^n + \tilde{\gamma}_2 t^{n-1} + \cdots + \tilde{\gamma}_{n+1} + o(t^{-p}).$$

Proof:

$$E[(N(t)+1)\cdots(N(t)+n)\,|\,A(t)=1] = \phi_{nc}(t)$$

$$= \frac{\tilde{E}e^{-\sigma\zeta_t} * \tilde{\phi}_n(t)}{\tilde{E}e^{-\sigma\zeta_t}} = \frac{\sum_{k=0}^{n} n_{(k)}\lambda_k\tilde{\phi}_{n-k}(t)+n!\Psi(t)}{\tilde{K}^*(\sigma)}$$

$$+ o(t^{-n-p})\{\sum_{k=0}^{n} n_{(k)}\lambda_k\tilde{\phi}_{n-k}(t)+n!\Psi(t)$$

by Lemmas 5.5, 5.8, and 5.9.

Define

$$\ell_k \equiv [\tilde{K}^*(\sigma)]^{-1}\lambda_k = \frac{\sigma\tilde{\mu}_1\lambda_k}{1-\omega} = \tilde{\gamma}_{k+1}.$$

Then

$$\phi_{nc}(t) = \sum_{k=0}^{n} n_{(k)}\ell_k\tilde{\phi}_{n-k}(t) + o(t^{-p}) \tag{5.2.21}$$

$$= \tilde{\gamma}_1 t^n + \tilde{\gamma}_2 t^{n-1} +\cdots+ \tilde{\gamma}_{n+1} + o(t^{-p})$$

by Lemma F. □

5.3 THE CONDITIONAL CUMULANTS OF N(t)

In the last section we learned that when $\tilde{\mu}_{n+p+1} < \infty$, $p \geq 0$, the nth conditional ϕ-moment of $N(t)$,

$$\phi_{nc}(t) = E[(N(t)+1)\cdots(N(t)+n)\,|\,A(t)=1],$$

is asymptotically an nth degree polynomial in t. Let

$$\Phi_{tc}(\zeta) = E[\frac{1}{(1-\zeta)^{N(t)+1}} \mid A(t)=1]$$

be the conditional ϕ-moment generating function. When $\tilde{\mu}_{n+p+1} < \infty$, we can write

$$\Phi_{tc}(\zeta) = 1 + \sum_{j=1}^{n} \phi_{jc}(t) \frac{\zeta^j}{j!} + o(\zeta^n) \text{ as } \zeta \to 0 \tag{5.3.1}$$

$$\log\Phi_{tc}(\zeta) \equiv \Psi_{tc}(\zeta) = \sum_{j=1}^{n} \psi_{jc}(t) \frac{\zeta^j}{j!} + o(\zeta^n) \text{ as } \zeta \to 0. \tag{5.3.2}$$

The coefficient $\psi_{nc}(t)$ is the nth conditional ψ-cumulant of $N(t)$; discovering its asymptotic form is our next task. We have that $\psi_{nc}(t)$ is the coefficient of $(n!)^{-1}\zeta^n$ in

$$\log\{1 + \sum_{j=1}^{n} \phi_{jc}(t) \frac{\zeta^j}{j!}\} ,$$

which, by equation (5.2.21), is

$$\log\{1 + \sum_{j=1}^{n} \frac{\zeta^j}{j!} \sum_{k=0}^{j} j_{(k)} \ell_k \tilde{\phi}_{j-k}(t) + \sum_{j=1}^{n} \frac{\zeta^j}{j!} \frac{r_j(t)}{(1+t)^{n+p-j}}\} \tag{5.3.3}$$

where $r_j(t) \to 0$ as $t \to \infty$.

Let

$$c_j \equiv \sum_{k=0}^{j} j_{(k)} \ell_k \tilde{\phi}_{j-k}(t).$$

From the Taylor expansion for $\log(1+z)$, we know $\psi_{nc}(t)$ is the coefficient of $(n!)^{-1}\zeta^n$ in

$$\sum_{j=1}^{n} c_j \frac{\zeta^j}{j!} + \sum_{j=1}^{n} \frac{r_j(t)}{(1+t)^{n+p-j}} \zeta^j$$

$$- \frac{1}{2} \{ \sum_{j=1}^{n} c_j \frac{\zeta^j}{j!} + \sum_{j=1}^{n} \frac{r_j(t)}{(1+t)^{n+p-j}} \zeta^j \}^2 + \cdots$$

$$+ (-1)^{n+1} \frac{1}{n} \{ \sum_{j=1}^{n} c_j \frac{\zeta^j}{j!} + \sum_{j=1}^{n} \frac{r_j(t)}{(1+t)^{n+p-j}} \zeta^j \}^n \tag{5.3.4}$$

$$= \sum_{j=1}^{n} c_j \frac{\zeta^j}{j!} - \frac{1}{2} \{ \sum_{j=1}^{n} c_j \frac{\zeta^j}{j!} \}^2 + \cdots + (-1)^{n+1} \frac{1}{n} \{ \sum_{j=1}^{n} c_j \frac{\zeta^j}{j!} \}^n$$

$$+ \frac{r(t)}{(1+t)^p} + \text{ terms involving mixtures of powers of}$$

$$\sum_{j=1}^{n} c_j \frac{\zeta^j}{j!} \text{ and } \sum_{j=1}^{n} \frac{r_j(t)}{(1+t)^{n+p-j}} \zeta^j . \tag{5.3.5}$$

We want to show that the coefficient of ζ^n in the mixture of powers can be represented by $\frac{r(t)}{(1+t)^p}$, $r(t) \to 0$ as $t \to \infty$. A typical summand in the mixture is

$$\{ \sum_{j=1}^{n} c_j \frac{\zeta^j}{j!} \}^m \{ \sum_{k=1}^{n} \frac{r_k(t)}{(1+t)^{n+p-k}} \zeta^k \}^q \tag{5.3.6}$$

where $m \geq 1$, $q \geq 1$, $m + q \leq n$. We now bound the absolute values of the coefficients of ζ^j and ζ^k individually. We know that $|c_j| < Mt^j$

and $|r_k(t)| < r(t)$ where $r(t) = o(1)$ as $t \to \infty$. Thus the coefficient of $(n!)^{-1}\zeta^n$ in (5.3.6) is smaller in absolute value than its coefficient in

$$M^m r(t)^q \{\sum_{j=1}^{n} (t\zeta)^j\}^m \{\sum_{k=1}^{n} \frac{1}{(1+t)^{n+1}} (\zeta(1+t))^k\}^q . \tag{5.3.7}$$

The coefficient of $(n!)^{-1}\zeta^n$ in (5.3.7) is

$$\frac{n!M^m r(t)^q}{(1+t)^{(n+p)q}} \{ t^m(1+t)^{n-m} + t^{m+1}(1+t)^{n-m-1} + \cdots + t^{n-q}(1+t)^q\}$$

$$= o(t^{-p}).$$

Therefore $\psi_{nc}(t)$ is the coefficient of $(n!)^{-1}\zeta^n$ in

$$\sum_{j=1}^{n} c_j \frac{\zeta^j}{j!} - \frac{1}{2}\{\sum_{j=1}^{n} c_j \frac{\zeta^j}{j!}\}^2 + \cdots + (-1)^{n+1}\frac{1}{n}\{\sum_{j=1}^{n} c_j \frac{\zeta^j}{j!}\}^n \tag{5.3.8}$$

plus a term of the form $o(t^{-p})$. But we recognize the coefficient of $(n!)^{-1}\zeta^n$ in (5.3.8) to be identical to its coefficient in

$$\log\{1 + \sum_{j=1}^{n} c_j \frac{\zeta^j}{j!}\}$$

$$= \log\{1 + \sum_{j=1}^{n} \frac{\zeta^j}{j!} \sum_{k=0}^{j} j_{(k)} \ell_k \tilde{\phi}_{j-k}(t)\}$$

$$= \log\{1 + \sum_{j=1}^{n} \zeta^j \sum_{k=0}^{j} \ell_k \frac{\tilde{\phi}_{j-k}(t)}{(j-k)!}\} . \tag{5.3.9}$$

Recall that $\ell_o = (\frac{\sigma\tilde{\mu}_1}{1-\omega})\lambda_o = 1$. Thus expression (5.3.9) is simply

$$\log\{\sum_{j=0}^{n} \zeta^j \sum_{k=0}^{j} \ell_k \frac{\tilde{\phi}_{j-k}(t)}{(j-k)!}\}. \tag{5.3.10}$$

Since $\sum_{k=0}^{j} \ell_k \frac{\tilde{\phi}_{j-k}(t)}{(j-k)!}$ is a term in the convolution sequence of $\{\ell_k\}_{k=0}^{n}$ and $\{\frac{\tilde{\phi}_k(t)}{k!}\}_{k=0}^{n}$, the coefficient we seek is the coefficient of $(n!)^{-1}\zeta^n$ in

$$\log\{[\sum_{j=0}^{n} \ell_j \zeta^j][\sum_{k=0}^{n} \frac{\tilde{\phi}_k(t)}{k!} \zeta^k]$$

$$= \log\{\sum_{j=0}^{n} \ell_j \zeta^j\} + \log\{\sum_{k=0}^{n} \frac{\tilde{\phi}_k(t)}{k!} \zeta^k\} . \tag{5.3.11}$$

We recognize

$$\sum_{k=0}^{n} \frac{\tilde{\phi}_k(t)}{k!} \zeta^k = 1 + \sum_{k=1}^{n} \frac{\tilde{\phi}_k(t)}{k!} \zeta^k$$

as the first n+1 terms in $\tilde{\Phi}_t(\zeta)$, the usual ϕ-moment generating function.

Hence

$$\psi_{nc}(t) = \eta_n + \tilde{\psi}_n(t) + o(t^{-p}) \tag{5.3.12}$$

where η_n is the coefficient of $(n!)^{-1}\zeta^n$ in

$$\log\{\sum_{j=0}^{n} \ell_j \zeta^j\} \tag{5.3.13}$$

and $\tilde{\psi}_n(t)$ is the <u>nth</u> ψ-cumulant of a proper renewal process whose lifetimes have distribution $\tilde{F}$. We can conclude $\eta_n = \tilde{\gamma}_{n+1}$ from (5.3.13).

<u>Theorem 5.3</u>

If n is a positive integer, $p \geq 0$, $\tilde{F} \in C$, and $\tilde{\mu}_{n+p+1} < \infty$, then the <u>nth</u> conditional cumulant of N(t) is

$$k_{nc}(t) = \tilde{k}_n(t) + \rho_n + o(t^{-p})$$

$$= \tilde{\gamma}_n t + \tilde{\gamma}_{n+1} + o(t^{-p}).$$

The constant ρ_n is $\begin{cases} \eta_1 \text{ if } n = 1 \\ \sum_{j=0}^{n-1} (-1)^j t_{n,n-j}\eta_{n-j} \text{ if } n \geq 2. \end{cases}$

Proof:

We know that

$$\psi_{nc}(t) = \eta_n + \tilde{\psi}_n(t) + o(t^{-p}).$$

The relationship cited in (5.2.4) between ψ-cumulants and conventional cumulants also holds between their conditional counterparts. That is,

$$k_{1c}(t) = \psi_{1c}(t) - 1 \tag{5.3.14}$$

$$k_{nc}(t) = \sum_{j=0}^{n-1} (-1)^j t_{n,n-j}\, \psi_{n-jc}(t),\ n \geq 2\ .$$

Hence

$$k_{1c}(t) = \{\tilde{\psi}_1(t)-1\} + \eta_1 + o(t^{-p}) = \tilde{k}_1(t) + \eta_1 + o(t^{-p}) \tag{5.3.15}$$

$$k_{nc}(t) = \sum_{j=0}^{n-1} (-1)^j t_{n,n-j}\tilde{\psi}_{n-j}(t) + \sum_{j=0}^{n-1} (-1)^j t_{n,n-j}\eta_{n-j} + o(t^{-p})$$

$$= \tilde{k}_n(t) + \rho_n + o(t^{-p}). \tag{5.3.16}$$

Finally, we appeal to Smith's 1959 theorem on the cumulants of renewal processes, which was quoted in Chapter 1. It ensures

$$\tilde{k}_n(t) = \tilde{\gamma}_n t + \tilde{\gamma}_{n+1} + o(t^{-p})$$

given our hypotheses. □

In order to derive the correction factor ρ_n of Theorem 5.3, we must find η_n, the coefficient of $(n!)^{-1}\zeta^n$ in

$$\log\{\sum_{j=0}^{n} \ell_j \zeta^j\}.$$

By definition, $\ell_j = \frac{\sigma\tilde{\mu}_1}{1-\omega}\lambda_j$ and thus

$$\ell_0 = 1$$

$$\ell_1 = \frac{1}{\sigma\tilde{\mu}_1} - \frac{1}{1-\omega} + \frac{\tilde{\mu}_2}{2\tilde{\mu}_1^2}$$

$$\ell_2 = \frac{1}{\sigma^2\tilde{\mu}_1^2} - \frac{1}{(1-\omega)\sigma\tilde{\mu}_1} - \frac{\tilde{\mu}_2}{2(1-\omega)\tilde{\mu}_1^2} + \frac{\tilde{\mu}_2}{\sigma\tilde{\mu}_1^3} - \frac{\tilde{\mu}_3}{6\tilde{\mu}_1^3} + \frac{\tilde{\mu}_2{}^2}{2\tilde{\mu}_1^4}$$

$$\ell_3 = \frac{1}{\sigma^3\tilde{\mu}_1^3} - \frac{1}{(1-\omega)\sigma^2\tilde{\mu}_1^2} - \frac{\tilde{\mu}_2}{(1-\omega)\sigma\tilde{\mu}_1^3} + \frac{3\tilde{\mu}_2}{2\sigma^2\tilde{\mu}_1^4} + \frac{\tilde{\mu}_3}{6(1-\omega)\tilde{\mu}_1^3}$$

$$- \frac{\tilde{\mu}_3}{3\sigma\tilde{\mu}_1^4} + \frac{\tilde{\mu}_4}{24\tilde{\mu}_1^4} - \frac{\tilde{\mu}_2{}^2}{2(1-\omega)\tilde{\mu}_1^4} + \frac{5\tilde{\mu}_2{}^2}{4\sigma\tilde{\mu}_1^5} - \frac{5\tilde{\mu}_2\tilde{\mu}_3}{12\tilde{\mu}_1^5} + \frac{5\tilde{\mu}_2{}^3}{8\tilde{\mu}_1^6} .$$

In general, $\ell_j = \tilde{\gamma}_{j+1}$.

Now

$$\log\{\sum_{j=0}^{n} \ell_j \zeta^j\} = \log\{1 + \sum_{j=1}^{n} \ell_j \zeta^j\}$$

$$= \sum_{j=1}^{n} \ell_j \zeta^j - \frac{1}{2}\{\sum_{j=1}^{n} \ell_j \zeta^j\}^2 + \frac{1}{3}\{\sum_{j=1}^{n} \ell_j \zeta^j\}^3 \cdots$$

Hence

$$\eta_1 = \ell_1$$

$$\eta_2 = 2\ell_2 - \ell_1{}^2$$

$$\eta_3 = 6\ell_3 - 6\ell_1\ell_2 + 2\ell_1{}^3$$

and, in general, $\eta_n = \tilde{\gamma}_{n+1}$.

Using the previously derived values of the ℓ's we have

$$\eta_1 = \frac{1}{\sigma\tilde{\mu}_1} - \frac{1}{1-\omega} + \frac{\tilde{\mu}_2}{2\tilde{\mu}_1^2}$$

$$\eta_2 = \frac{1}{\sigma^2\tilde{\mu}_1^2} + \frac{\tilde{\mu}_2}{\sigma\tilde{\mu}_1^3} - \frac{\tilde{\mu}_3}{3\tilde{\mu}_1^3} + \frac{3\tilde{\mu}_2^2}{4\tilde{\mu}_1^4} - \frac{1}{(1-\omega)^2}$$

$$\eta_3 = \frac{2}{\sigma^3\tilde{\mu}_1^3} - \frac{2}{(1-\omega)^3} + \frac{3\tilde{\mu}_2}{\sigma^2\tilde{\mu}_1^4} - \frac{\tilde{\mu}_3}{\sigma\tilde{\mu}_1^4} + \frac{\tilde{\mu}_4}{4\tilde{\mu}_1^4}$$

$$+ \frac{3\tilde{\mu}_2^2}{\sigma\tilde{\mu}_1^5} - \frac{2\tilde{\mu}_2\tilde{\mu}_3}{\tilde{\mu}_1^5} + \frac{5\tilde{\mu}_2{}^3}{2\tilde{\mu}_1^6} .$$

<u>Corollary 5.1</u>

If $\tilde{\mu}_{2+p} < \infty$, $p \geq 0$ and if $\tilde{F} \in C$, then

$$E[N(t)|A(t)=1] = \frac{t}{\tilde{\mu}_1} + \frac{\tilde{\mu}_2}{\tilde{\mu}_1^2} + \frac{1}{\sigma\tilde{\mu}_1} - \frac{2-\omega}{1-\omega} + o(t^{-p}).$$

Proof:

$$E[N(t)|A(t)=1] = k_{1c}(t) = \tilde{k}_1(t) + \eta_1 + o(t^{-p})$$

$$\tilde{k}_1(t) = \tilde{H}(t) = \frac{t}{\tilde{\mu}_1} + \frac{\tilde{\mu}_2}{2\tilde{\mu}_1^2} - 1 + o(t^{-p})$$

by Smith (1959) and thus the result follows. Note that this formula agrees with the one derived in section 5.1 by different methods.

□

<u>Corollary 5.2</u>

If $\tilde{\mu}_{3+p} < \infty$, $p \geq 0$ and if $\tilde{F} \in C$, then

$\mathrm{Var}[N(t)|A(t) = 1]$

$$= \left(\frac{\tilde{\mu}_2-\tilde{\mu}_1^2}{\tilde{\mu}_1^3}\right)t + \frac{2\tilde{\mu}_2^2}{\tilde{\mu}_1^4} - \frac{\tilde{\mu}_3}{\tilde{\mu}_1^3} - \frac{\tilde{\mu}_2}{\tilde{\mu}_1^2} + \frac{\tilde{\mu}_2}{\sigma\tilde{\mu}_1^3} + \frac{1}{\sigma^2\tilde{\mu}_1^2} - \frac{1}{\sigma\tilde{\mu}_1} - \frac{\omega}{(1-\omega)^2} + o(t^{-p}).$$

Proof:

$$\mathrm{Var}[N(t)|A(t)=1] = k_{2c}(t) = \tilde{k}_2(t) + \rho_2 + o(t^{-p})$$

$$\rho_2 = \eta_2-\eta_1 = \frac{3\tilde{\mu}_2^2}{4\tilde{\mu}_1^4} - \frac{\tilde{\mu}_3}{3\tilde{\mu}_1^3} - \frac{\tilde{\mu}_2}{2\tilde{\mu}_1^2} + \frac{\tilde{\mu}_2}{\sigma\tilde{\mu}_1^3} + \frac{1}{\sigma^2\tilde{\mu}_1^2} - \frac{1}{\sigma\tilde{\mu}_1} - \frac{\omega}{(1-\omega)^2}$$

and by Smith (1959),

$$\tilde{k}_2(t) = \left(\frac{\tilde{\mu}_2-\tilde{\mu}_1^2}{\tilde{\mu}_1^3}\right)t + \frac{5\tilde{\mu}_2^2}{4\tilde{\mu}_1^4} - \frac{2\tilde{\mu}_3}{3\tilde{\mu}_1^3} - \frac{\tilde{\mu}_2}{2\tilde{\mu}_1^2} + o(t^{-p}).$$

The result follows and agrees with the independently derived equation in (5.2.28). □

If we are interested in higher cumulants we can proceed in this same way. For example,

$$k_{3c}(t) = \tilde{k}_3(t) + \eta_3 - 3\eta_2 + \eta_1 + o(t^{-p})$$

whenever $\tilde{\mu}_{4+p} < \infty$ and $\tilde{F} \in C$.

The form for $\tilde{k}_3(t)$ can be found by the methods outlined in Smith (1959) and the correction factor has already been derived. We find

$$k_{3c}(t) = \{(\frac{3\tilde{\mu}_2^2}{\tilde{\mu}_1^5} - \frac{\tilde{\mu}_3}{\tilde{\mu}_1^4} - \frac{3\tilde{\mu}_2}{\tilde{\mu}_1^3} - \frac{1}{\tilde{\mu}_1})t + \frac{3\tilde{\mu}_4}{4\tilde{\mu}_1^4} - \frac{5\tilde{\mu}_2\tilde{\mu}_3}{\tilde{\mu}_1^5}$$

$$+ \frac{11\tilde{\mu}_2^3}{2\tilde{\mu}_1^6} + \frac{2\tilde{\mu}_3}{\tilde{\mu}_1^3} - \frac{15\tilde{\mu}_2^2}{4\tilde{\mu}_1^4} + \frac{\tilde{\mu}_2}{2\tilde{\mu}_1^2}\}$$

$$+\{\frac{\tilde{\mu}_4}{4\tilde{\mu}_1^4} - \frac{2\tilde{\mu}_2\tilde{\mu}_3}{\tilde{\mu}_1^5} + \frac{5\tilde{\mu}_2^3}{2\tilde{\mu}_1^6} + \frac{\tilde{\mu}_3}{\tilde{\mu}_1^3} - \frac{9\tilde{\mu}_2^2}{4\tilde{\mu}_1^4} + \frac{\tilde{\mu}_2}{2\tilde{\mu}_1^2} +$$

$$\frac{2}{\sigma^3\tilde{\mu}_1^3} - \frac{2}{(1-\omega)^3} + \frac{3\tilde{\mu}_2}{\sigma^2\tilde{\mu}_1^4} - \frac{\tilde{\mu}_3}{\sigma\tilde{\mu}_1^4} + \frac{3\tilde{\mu}_2^2}{\sigma\tilde{\mu}_1^5} - \frac{3}{\sigma^2\tilde{\mu}_1^2} - \frac{3\tilde{\mu}_2}{\sigma\tilde{\mu}_1^3}$$

$$+ \frac{3}{(1-\omega)^2} + \frac{1}{\sigma\tilde{\mu}_1} - \frac{1}{1-\omega}\} + o(t^{-p}).$$

That is,

$$k_{3c}(t) = (\frac{3\tilde{\mu}_2^2}{\tilde{\mu}_1^5} - \frac{\tilde{\mu}_3}{\tilde{\mu}_1^4} - \frac{3\tilde{\mu}_2}{\tilde{\mu}_1^3} - \frac{1}{\tilde{\mu}_1})t$$

$$+ \frac{\tilde{\mu}_4}{\tilde{\mu}_1^4} - \frac{7\tilde{\mu}_2\tilde{\mu}_3}{\tilde{\mu}_1^5} + \frac{8\tilde{\mu}_2^3}{\tilde{\mu}_1^6} + \frac{3\tilde{\mu}_3}{\tilde{\mu}_1^3} - \frac{6\tilde{\mu}_2^2}{\tilde{\mu}_1^4} + \frac{\tilde{\mu}_2}{\tilde{\mu}_1^2} + \frac{2}{\sigma^3\tilde{\mu}_1^3} - \frac{2}{(1-\omega)^3}$$

$$+ \frac{3\tilde{\mu}_2}{\sigma^2\tilde{\mu}_1^4} - \frac{\tilde{\mu}_3}{\sigma\tilde{\mu}_1^4} + \frac{3\tilde{\mu}_2^2}{\sigma\tilde{\mu}_1^5} - \frac{3}{\sigma^2\tilde{\mu}_1^2} - \frac{3\tilde{\mu}_2}{\sigma\tilde{\mu}_1^3} + \frac{3}{(1-\omega)^2} + \frac{1}{\sigma\tilde{\mu}_1} - \frac{1}{1-\omega} + o(t^{-p}).$$

(5.3.17)

We will confirm this last formula in Chapter 6.

CHAPTER 6

THE CONDITIONAL CUMULANTS OF CUMULATIVE PROCESSES

6.1 INTRODUCTION

We want to investigate $E[W(t)^n|A(t)=1]$ and the conditional cumulants of the transient cumulative process $W(t)$. We know that

$$E[W(t)^n|A(t)=1] = \frac{\tilde{E}e^{-\sigma\zeta_t}W(t)^n}{\tilde{E}e^{-\sigma\zeta_t}} .$$

In the last chapter we were concerned with the special (Type B) cumulative process

$$W(t) \equiv N(t) = \sum_{j=1}^{N(t)} 1$$

and found that in this case

$$\tilde{E}e^{-\sigma\zeta_t}W(t)^n = \tilde{E}e^{-\sigma\zeta_t}*\tilde{E}W(t)^n.$$

The fact that $\tilde{E}e^{-\sigma\zeta_t}W(t)^n$ is the Stieltjes convolution of two familiar functions explains the success of our Laplace Transform attack.

In this chapter *we shall restrict ourselves to general Type B processes*

$$W(t) = \sum_{j=1}^{N(t)} Y_j ,$$

for we shall find that whenever $E|Y_1|^n < \infty$,

$$\tilde{E}e^{-\sigma\zeta_t}W(t)^n = \tilde{E}e^{-\sigma\zeta_t} * \tilde{E}W(t)^n$$

and the Laplace Transform is again effective.

Suppose W(t) is built from the iid sequence (X_1, Y_1), (X_2, Y_2), ... where G(x,y) is the joint defective distribution of (X_1, Y_1). Let $F(x) = G(x,\infty)$ and assume $F(\infty) = \omega < 1$. As in the last two chapters, we assume there is some $\sigma > 0$ making

$$\tilde{F}(x) = \int_0^x e^{\sigma u}\, dF(u)$$

and

$$\tilde{G}(x,y) = \int_0^x \int_{-\infty}^y e^{\sigma u} G(du,dv)$$

proper distributions. Define $\tilde{G}^*(s,\theta) \equiv \tilde{E}e^{-sX_1+i\theta Y_1}$. Let $\tilde{\mu}_{rs} = \tilde{E}X_1^r Y_1^s$, $\tilde{\mu}_r = \tilde{\mu}_{r0} = \tilde{E}X_1^r$, and $\tilde{\kappa}_s = \tilde{\mu}_{0s} = \tilde{E}Y_1^s$.

<u>Lemma 6.1</u>

If $\tilde{E}|Y_1|^n < \infty$, then

$\tilde{E}e^{-\sigma\zeta_t}W(t)^n = \tilde{E}e^{-\sigma\zeta_t} * \tilde{E}W(t)^n$ whenever W(t) is a Type B cumulative process.

Proof:

$$L^0\{\tilde{E}e^{-\sigma\zeta_t+i\theta W(t)}\} = \int_0^\infty e^{-st}\, \tilde{E}e^{-\sigma(S_{N(t)+1}-t)+i\theta\sum_{j=1}^{N(t)} Y_j}\, dt.$$

We can reverse the order of integration to find

$$= \tilde{E}\int_0^\infty e^{-(s-\sigma)t}\, e^{-\sigma S_{N(t)+1}+i\theta\sum_{j=1}^{N(t)} Y_j}\, dt$$

$$= \tilde{E} \sum_{n=0}^{\infty} \int_{S_n}^{S_{n+1}} e^{-(s-\sigma)t} e^{-\sigma S_{n+1} + i\theta \sum_{j=1}^{n} Y_j} dt$$

$$= \tilde{E} \sum_{n=0}^{\infty} e^{-\sigma S_{n+1} + i\theta \sum_{j=1}^{n} Y_j} \int_{S_n}^{S_{n+1}} e^{-(s-\sigma)t} dt$$

$$= (s-\sigma)^{-1} \tilde{E} \sum_{n=0}^{\infty} e^{-\sigma S_{n+1} + i\theta \sum_{j=1}^{n} Y_j} e^{-(s-\sigma) S_n} (1-e^{-(s-\sigma) X_{n+1}})$$

$$= (s-\sigma)^{-1} \tilde{E} \sum_{n=0}^{\infty} e^{-sS_n + i\theta \sum_{j=1}^{n} Y_j} [e^{-\sigma X_{n+1}} - e^{-sX_{n+1}}]$$

$$= (s-\sigma)^{-1} \sum_{n=0}^{\infty} \tilde{E}[e^{-sS_n + i\theta \sum_{j=1}^{n} Y_j}] \tilde{E}[e^{-\sigma X_{n+1}} - e^{-sX_{n+1}}]$$

since (X_n, Y_n) is independent of (X_m, Y_m) for $n \neq m$.

$$= (s-\sigma)^{-1} [\tilde{F}^*(\sigma) - \tilde{F}^*(s)] \sum_{n=0}^{\infty} \tilde{G}^*(s,\theta)^n$$

$$= \frac{\omega - F^*(s)}{(s-\sigma)(1-\tilde{G}^*(s,\theta))} \, . \tag{6.1.1}$$

Next we consider

$$L^0\{\tilde{E}e^{i\theta W(t)}\} = \int_0^{\infty} e^{-st} \tilde{E} e^{i\theta \sum_{j=1}^{N(t)} Y_j} dt$$

$$= \tilde{E} \sum_{n=0}^{\infty} \int_{S_n}^{S_{n+1}} e^{-st} e^{i\theta \sum_{j=1}^{n} Y_j} dt = \tilde{E} \sum_{n=0}^{\infty} e^{i\theta \sum_{j=1}^{n} Y_j} \int_{S_n}^{S_{n+1}} e^{-st} dt$$

$$= s^{-1} \tilde{E} \sum_{n=0}^{\infty} e^{-sS_n + i\theta \sum_{j=1}^{n} Y_j} (1 - e^{-sX_{n+1}})$$

$$= s^{-1} \sum_{n=0}^{\infty} \tilde{E} e^{-sS_n + i\theta \sum_{j=1}^{n} Y_j} \tilde{E}(1 - e^{-sX_{n+1}})$$

$$= \frac{1 - \tilde{F}^*(s)}{s[1 - \tilde{G}^*(s,\theta)]} \quad . \tag{6.1.2}$$

From Lemma 5.6 we recall

$$L\{\tilde{E} e^{-\sigma\zeta_t}\} = \frac{s[\omega - \tilde{F}^*(s)]}{(s-\sigma)[1 - \tilde{F}^*(s)]}$$

and therefore

$$L^0\{\tilde{E} e^{-\sigma\zeta_t + i\theta W(t)}\} = L\{\tilde{E} e^{-\sigma\zeta_t}\} L^0\{\tilde{E} e^{i\theta W(t)}\} \; . \tag{6.1.3}$$

If $E|Y_1|^n < \infty$, we can conclude

$$L^0\{\tilde{E} e^{-\sigma\zeta_t} W(t)^n\} = L\{\tilde{E} e^{-\sigma\zeta_t}\} L^0\{\tilde{E} W(t)^n\} \; . \tag{6.1.4}$$

$\tilde{E}W(t)^n$ is of bounded variation in every finite interval. Hence its Laplace-Stieltjes Transform exists and we have

$$L\{\tilde{E}e^{-\sigma\zeta_t}W(t)^n\} = sL^0\{\tilde{E}e^{-\sigma\zeta_t}W(t)^n\} = sL\{\tilde{E}e^{-\sigma\zeta_t}\}L^0\{\tilde{E}W(t)^n\}$$

$$= L\{\tilde{E}e^{-\sigma\zeta_t}\}\, L\{\tilde{E}W(t)^n\} = L\{\tilde{E}e^{-\sigma\zeta_t}*\tilde{E}W(t)^n\} . \qquad (6.1.5)$$

By the uniqueness theorem for Laplace Transforms we conclude

$$\tilde{E}e^{-\sigma\zeta_t}W(t)^n = \tilde{E}e^{-\sigma\zeta_t}*\tilde{E}W(t)^n. \qquad \square$$

To study the properties of $E[W(t)^n|A(t) = 1]$, we study the Laplace Transform of its numerator, which we have just found to be $sL\{\tilde{E}e^{-\sigma\zeta_t}\}\, L^0\{\tilde{E}W(t)^n\}$. We examined $L\{\tilde{E}e^{-\sigma\zeta_t}\}$ in the last chapter and thus focus our attention on $L^0\{\tilde{E}W(t)^n\}$. A special notation for $W(t)^n$ will help us calculate its transform.

$$W(t) = \sum_{j=1}^{\infty} Z_j(t)Y_j \text{ where } Z_j(t) = \begin{cases} 1 \text{ if } S_j \le t \\ 0 \text{ otherwise} \end{cases}$$

$$W(t)^n = [\sum_{j=1}^{\infty} Z_j(t)Y_j]^n.$$

Consider the particular case n = 3.

$$W(t)^3 = [\sum_{j=1}^{\infty} Z_j(t)Y_j^2 + 2\sum_{j>k\ge 1} Z_j(t)Y_jY_k][\sum_{\ell=1}^{\infty} Z_\ell(t)Y_\ell]$$

$$= \sum_{j=1}^{\infty} Z_j(t)Y_j^3 + 3\sum_{j>k\ge 1} Z_j(t)Y_j^2Y_k + 3\sum_{j>k\ge 1} Z_j(t)Y_jY_k^2$$

$$+ 6\sum_{j>k\ge 1} Z_j(t)Y_jY_kY_\ell$$

$$= \textstyle\sum_t(3) + 3\sum_t(2,1) + 3\sum_t(1,2) + 6\sum(1,1,1).$$

In general,

$$W(t)^n = \textstyle\sum_t(n) + n\sum_t(n-1,1) + \cdots + n!\sum_t(1,1,\cdots,1). \qquad (6.1.6)$$

A typical summand in (6.1.6) is

$$c\textstyle\sum_t(p_1,\cdots,p_{1c}) = c \displaystyle\sum_{r_1>r_2>\cdots>r_k\geq 1} Z_{r_1}(t)Y_{r_1}^{p_1}Y_{r_2}^{p_2}\cdots Y_{r_k}^{p_k}$$

where $\sum_{j=1}^{k} p_j = n$ and c is a constant depending upon $(p_1,\cdots,p_k)$.

This notation is adapted from notation Smith developed for Type A processes in his 1979 technical report.

Let $S(p_1,\cdots,p_k) \equiv L^0\{\tilde{E}\sum_t(p_1,\cdots,p_k)\}$. (6.1.7)

Thus

$$L^0\{\tilde{E}W(t)^n\} = S(n) + nS(n-1,1) + \cdots + n!S(1,\cdots 1). \qquad (6.1.8)$$

Consider the term $S(p_1,\cdots,p_k)$. Let $q = \max(p_1,\cdots,p_k)$ and assume $E|Y_1|^q < \infty$.

$$S(p_1,\cdots p_k) = \int_0^\infty e^{-st}\,\tilde{E}\textstyle\sum_t(p_1,\cdots,p_k)dt$$

$$= \tilde{E}\int_0^\infty e^{-st}\textstyle\sum_t(p_1,\cdots,p_k)dt$$

$$= \tilde{E} \sum_{r_1 > \cdots > r_k \geq 1} \int_0^\infty e^{-st} Z_{r_1}(t) Y_{r_1}^{p_1} Y_{r_2}^{p_2} \cdots Y_{r_k}^{p_k} \, dt$$

$$= \tilde{E} \sum_{r_1 > \cdots > r_k \geq 1} Y_{r_1}^{p_1} \cdots Y_{r_k}^{p_k} \int_0^\infty e^{-st} Z_{r_1}(t) dt$$

$$= \tilde{E} \sum_{r_1 > \cdots > r_k \geq 1} Y_{r_1}^{p_1} \cdots Y_{r_k}^{p_k} \int_{S_{r_1}}^\infty e^{-st} dt$$

$$= s^{-1} \sum_{r_1 > \cdots > r_k \geq 1} (\tilde{E} Y_{r_1}^{p_1} e^{-sX_{r_1}}) \cdots (\tilde{E} Y_{r_k}^{p_k} e^{-sX_{r_k}}) (\tilde{F}^*(s))^{r_1 - k}. \tag{6.1.9}$$

We could interchange the order of integration in the course of these manipulations because if we replace each term $Y_{r_j}^{p_j}$ by its absolute value, the resulting k-fold sum is convergent.

Define

$$C_k(s) \equiv \left(\frac{\partial}{i\partial\theta}\right)^k \tilde{G}^*(s,\theta)\bigg|_{\theta=0} = \tilde{E} Y_j^k e^{-sX_j} \quad j=1,2,\cdots$$

Then

$$S(p_1,\cdots,p_k) = C_{p_1}(s) \cdots C_{p_k}(s) s^{-1} \sum_{r_1=k}^{\infty} \tilde{F}^*(s)^{r_1-k} \left\{ \sum_{r_2=k-1}^{r_1-1} \cdots \sum_{r_k=1}^{r_{k-1}-1} 1 \right\}$$

$$= C_{p_1}(s) \cdots C_{p_k}(s) s^{-1} \sum_{r_1=k}^{\infty} \tilde{F}^*(s)^{r_1-k} (r_1-1) \cdots (r_1-k+1) [(k-1)!]^{-1}$$

$$= \frac{C_{p_1}(s) \cdots C_{p_k}(s)}{s(k-1)!} \sum_{r_1=k-1}^{\infty} \tilde{F}^*(s)^{r_1-k+1} \, r_{1(k-1)}$$

$$= \frac{C_{p_1}(s)\cdots C_{p_k}(s)}{s[1-\tilde{F}^*(s)]^k} \quad . \tag{6.1.10}$$

Therefore $L\{\tilde{E}e^{-\sigma\zeta_t}W(t)^n\}$ is composed of terms like

$$\frac{C_{p_1}(s)\cdots C_{p_k}(s)}{Q(s)^k} \quad \frac{s[\omega-\tilde{F}^*(s)]}{(s-\sigma)[1-\tilde{F}^*(s)]} \quad . \tag{6.1.11}$$

If we assume $\tilde{E}|Y_1|^q < \infty$, $\tilde{\mu}_{k+p+1} < \infty$, $p \geq 0$, and $\tilde{E}X_1^{k+p}|Y_1|^q < \infty$ where $q = \max(p_1,\cdots,p_k)$, what are the properties of this term?

In Chapter 5 we found that when $\tilde{\mu}_{k+p+1} < \infty$,

$$\frac{s[\omega-\tilde{F}^*(s)]}{(s-\sigma)[1-\tilde{F}^*(s)]} = \lambda_0 + \lambda_1 Q + \cdots + \lambda_k Q^k + R_k^*(s)$$

where $R_k(t)$ is a function of bounded variation on $[0,\infty)$ having k+p absolute moments, and $R^*(s) = o(|s|^k)$ as $|s| \to 0$. If we write

$$B^*(s) = \frac{s[\omega-\tilde{F}^*(s)]}{(s-\sigma)[1-\tilde{F}^*(s)]}$$

then $\lambda_0 = B^*(0) = \tilde{\gamma}_1$ and for $j \geq 1$ λ_j is a rational function of the first j moments of B(x) and $\tilde{F}(x)$. As the j<u>th</u> moment of B(x) is a rational function of $\tilde{\mu}_1,\cdots,\tilde{\mu}_{j+1}$, $\lambda_j = \tilde{\gamma}_{j+1}$, $j \geq 0$.

We now express $C_{p_r}(s)$ in powers of Q.

$$C_{p_r}(s) = \tilde{E}Y_1^{p_r}e^{-sX_1} = \int_0^\infty e^{-sx}\int_{-\infty}^\infty y^{p_r}\tilde{G}(dx,dy)$$

$$= L\{\int_{-\infty}^\infty y^{p_r}\tilde{G}(x,dy)\} \equiv L\{C_{p_r}(x)\} \quad . \tag{6.1.12}$$

The assumption $EX_1^{k+p}|Y_1|^q < \infty$ implies $C_{p_r}(x)$ is a function of bounded variation on $[0,\infty)$ having k+p absolute moments. Therefore Smith's Lemma D guarantees

$$C_{p_r}(s) = \tilde{\mu}_{0p_r} - \tilde{\mu}_{1p_r} s + \tilde{\mu}_{2p_r} \frac{s^2}{2!} \cdots + \frac{\tilde{\mu}_{kp_r}(-s)^k}{k!} C_{p_r(k)}(s) \tag{6.1.13}$$

where $C_{p_r(k)}(x)$ is a function of bounded variation on $[0,\infty)$ having p absolute moments.

By the same argument applied to

$$B^*(s) = \frac{s[\omega-\tilde{F}^*(s)]}{(s-\sigma)[1-\tilde{F}^*(s)]},$$

we can rewrite $C_{p_r}(s)$ in powers of $Q(s)$.

$$C_{p_r}(s) = \tilde{\mu}_{0p_r} + \tilde{\gamma}_{1p_r} Q + \cdots + \tilde{\gamma}_{kp_r} Q^k + D^*_{p_r}(s) \tag{6.1.14}$$

where $D_{p_r}(t)$ is a function of bounded variation on $[0,\infty)$ having k+p absolute moments and $D^*_{p_r}(s) = o(|s|^k)$ as $|s| \to 0$. The coefficient $\tilde{\gamma}_{jp_r}$ is a rational function of $\tilde{\mu}_{0p_r}$ and the first j moments of $C_{p_r}(x)$ and $\tilde{F}(x)$. That is, $\tilde{\gamma}_{jp_r}$ is a rational function of $\tilde{\kappa}_{p_r}$, $\tilde{\mu}_1, \cdots, \tilde{\mu}_j$, and $\tilde{\mu}_{1p_r}, \cdots, \tilde{\mu}_{jp_r}$.

Expression (6.1.11) consists, in part, of $C_{p_1}(s) \cdots C_{p_k}(s)$

$$= \tilde{\gamma}_{0q} + \tilde{\gamma}_{1q} Q + \cdots + \tilde{\gamma}_{kq} Q^k + D^*_k(s). \tag{6.1.15}$$

As usual, $D_k(t)$ is a function of bounded variation on $[0,\infty)$ having k+p absolute moment; $D^*_k(s) = o(|s|^k)$ as $|s| \to 0$.

Hence

$$\frac{C_{p_1}(s)\cdots C_{p_k}(s)}{Q(s)^k} \cdot \frac{s[\omega-\tilde{F}^*(s)]}{(s-\sigma)[1-\tilde{F}^*(s)]}$$

$$= \frac{\tilde{\gamma}_{0q}}{Q^k}\ [\lambda_0+\lambda_1 Q+\cdots+\ \lambda_k Q^k\ +\ R_k{}^*(s)]$$

$$+\ \frac{\tilde{\gamma}_{1q}}{Q^{k-1}}\ [\lambda_0+\lambda_1 Q+\cdots+\ \lambda_k Q^k\ +\ R_k{}^*(s)]$$

$$+\cdots+\ \tilde{\gamma}_{kq}[\lambda_0+\lambda_1 Q+\cdots+\ \lambda_k Q^k\ +\ R_k{}^*(s)]$$

$$+\ D_k{}^*(s)[\lambda_0+\lambda_1 Q\ +\cdots+\ \lambda_k Q^k\ +\ R_k{}^*(s)]$$

$$= \frac{\delta_1}{Q^k} + \frac{\delta_2}{Q^{k-1}} +\cdots+\ \delta_{k+1}\ +\ \frac{R^*(s)}{Q^k} \tag{6.1.16}$$

where $R(t)$ is a function of bounded variation on $[0,\infty)$ having $k+p$ absolute moments. In addition, $R^*(s) = o(|s|^k)$ as $|s| \to 0$. The coefficients $\delta_1,\cdots,\ \delta_{k+1}$ are rational functions of $\tilde{\mu}_1,\cdots,\ \tilde{\mu}_{k+1}$, $\tilde{\kappa}_1,\cdots,\ \tilde{\kappa}_q$, and $\tilde{\mu}_{rs}$ for $r = 1,\cdots,\ k$ and $s = 1,\cdots,\ q$.

As in the last chapter, this means that

$$L^{-1}\left\{ \frac{C_{p_1}(s)\cdots C_{p_k}(s)}{Q(s)^k} \cdot \frac{s[\omega-\tilde{F}^*(s)]}{(s-\sigma)[1-\tilde{F}^*(s)]} \right\}$$

$$= \delta_1\tilde{\phi}_k(t) + \delta_2\tilde{\phi}_{k-1}(t)+\cdots+ \delta_{k+1} + \Psi(t). \qquad (6.1.17)$$

The function $\Psi(t)$ can be expressed as

$$\Psi(t) = \frac{\lambda(t)}{(1+t)^p}$$

where $\lambda(t)$ is of bounded variation, tends to zero as t approaches infinity, and if $p \geq 1$, $\frac{\lambda(t)}{1+t}$ belongs to the class L_1.

Thus far we have focussed our attention on just one of the summands in $\tilde{E}e^{-\sigma\zeta_t}W(t)^n$. If we are to make statements similar to (6.1.17) about all the terms making up $\tilde{E}e^{-\sigma\zeta_t}W(t)^n$, we must broaden our assumptions. Equation (6.1.17) depends on the suppositions:

1) $\tilde{\mu}_{k+p+1} < \infty$

2) $\tilde{E}|Y_1|^q < \infty$

3) $\tilde{E}X_1^{k+p}|Y_1|^q < \infty$.

We need 1, 2, and 3 to hold for all possible partitions $(p_1,\cdots, p_k)$ of n. The largest possible value of q arises when $k = 1$ and $q = p_1 = n$. Thus we must assume $\tilde{E}|Y_1|^n < \infty$ and $\tilde{\mu}_{n+p+1} < \infty$. As for the assumption that $\tilde{E}X_1^{k+p}|Y_1|^q < \infty$, note that $q \leq n+1-k$. Hence we require $\tilde{E}X_1^{r+p}|Y_1|^s < \infty$ for $r+s \leq n+1$, $s \leq n$, and $r \leq n$.

<u>Lemma 6.2</u>

Let n be a positive integer and $p \geq 0$. Suppose $\tilde{\mu}_{n+p+1} < \infty$, $\tilde{E}|Y_1|^n < \infty$, and $\tilde{E}X_1^{r+p}|Y_1|^s < \infty$ for $r + s \leq n + 1$, $s \leq n$, and $r \leq n$. Then

$$\tilde{E}e^{-\sigma\zeta_t}W(t)^n = \delta_1\tilde{\phi}_n(t) + \delta_2\tilde{\phi}_{n-1}(t) + \cdots + \delta_{n+1} + \frac{\lambda(t)}{(1+t)^p}$$

where $\lambda(t)$ is of bounded variation, tends to zero as t approaches infinity, and if $p \geq 1$, $\frac{\lambda(t)}{1+t}$ belongs to L_1.

The coefficients $\delta_1, \cdots, \delta_{n+1}$ are rational functions of $\tilde{\mu}_1, \cdots, \tilde{\mu}_{n+1}, \tilde{\kappa}_1, \cdots, \tilde{\kappa}_n$, and $\tilde{\mu}_{rs}$ for $r + s \leq n + 1$, $r \leq n$, and $s \leq n$.

<u>Theorem 6.1</u>

Suppose $\tilde{F} \in \mathcal{C}$, n is a positive integer, and $p \geq 0$. If $\tilde{\mu}_{n+p+1} < \infty$, $\tilde{E}|Y_1|^n < \infty$, and $\tilde{E}X_1^{r+p}|Y_1|^s < \infty$ for $r + s \leq n + 1$, $r \leq n$, and $s \leq n$, then

$$E[W(t)^n | A(t) = 1]$$

$$= \delta_1 t^n + \delta_2 t^{n-1} + \cdots + \delta_{n+1} + o(t^{-p}).$$

The coefficients $\delta_1, \delta_2, \cdots, \delta_{n+1}$ are rational functions of $\tilde{\mu}_1, \cdots, \tilde{\mu}_{n+1}, \tilde{\kappa}_1, \cdots, \tilde{\kappa}_n$, and the product moments $\tilde{\mu}_{rs}$ for $r + s \leq n + 1$, $r \leq n$, and $s \leq n$.

Proof:

$$E[W(t)^n | A(t)=1] = \frac{\tilde{E}e^{-\sigma\zeta_t}W(t)^n}{\tilde{E}e^{-\sigma\zeta_t}}$$

$$= \{\sum_{k=1}^{n+1} \delta_k \tilde{\phi}_{n+1-k}(t) + \frac{\lambda(t)}{(1+t)^p}\} \{\frac{\sigma\tilde{\mu}_1}{1-\omega} + o(t^{-n-p})\} \qquad (6.1.18)$$

by Lemma 5.5 and Lemma 6.2.

Since $\tilde{F} \in C$ and $\tilde{\mu}_{n+p+1} < \infty$, the conditions of Lemma F are fulfilled. Hence

$$\tilde{\phi}_r(t) = \tilde{\gamma}_1 t^r + \tilde{\gamma}_2 t^{r-1} + \cdots + \tilde{\gamma}_{r+1} + o(t^{-n+r-p}) \qquad (6.1.19)$$

for $1 \le r \le n$. The result follows easily from (6.1.18) and (6.1.19)

□

6.2 THE CONDITIONAL CUMULANTS OF W(t)

We have learned that under certain conditions on the moments and product moments of (X_1, Y_1), $E[W(t)^n | A(t)=1]$ is asymptotically an nth degree polynomial in t. We now turn from the conditional moments to the conditional cumulants of W(t).

Define

$$M_{\theta c}(t) \equiv E[e^{i\theta W(t)} | A(t) = 1]. \qquad (6.2.1)$$

$M_{\theta c}(t)$ is the conditional moment generating function of W(t). If $E|Y_1|^n < \infty$,

$$M_{\theta c}(t) = 1 + \sum_{j=1}^{n} m_{jc}(t) \frac{(i\theta)^j}{j!} + o(\theta^n) \text{ as } \theta \to 0 \qquad (6.2.2)$$

where $m_{jc}(t) = E[W(t)^j | A(t)=1]$.

Let

$$K_{\theta c}(t) \equiv \log M_{\theta c}(t)$$

$$= \sum_{j=1}^{n} k_{jc}(t) \frac{(i\theta)^j}{j!} + o(\theta^n) \text{ as } \theta \to 0. \qquad (6.2.3)$$

$K_{\theta c}(t)$ is the conditional cumulant generating function of $W(t)$. The n<u>th</u> conditional cumulant is

$$k_{nc}(t) = \sum{}^{*} C(p_1,\cdots,p_k) m_{p_1 c}(t) \cdots m_{p_k c}(t) \qquad (6.2.4)$$

where the summation is over all partitions $(p_1,\cdots,p_k)$ of n and where $C(p_1,\cdots,p_k)$ is a constant.

Under the conditions of Theorem 6.1,

$$m_{p_r c}(t) = \delta_1 t^{p_r} + \delta_2 t^{p_r - 1} + \cdots + \delta_{p_r+1} + o(t^{-n+p_r-p})$$

for $p_r = 1, 2, \cdots, n$. Hence

$$k_{nc}(t) = \delta_1 t^n + \delta_2 t^{n-1} + \cdots + \delta_{n+1} + o(t^{-p}). \qquad (6.2.5)$$

As in Theorem 6.1, the coefficients $\delta_1, \cdots, \delta_{n+1}$ are rational functions of certain moments and product moments of (X_1, Y_1); they do not depend on the particular form of the distribution $\tilde{G}(x,y)$. While (6.2.5) is valid whenever $\tilde{\mu}_{n+p+1} < \infty$, $E|Y_1|^n < \infty$, and $EX_1^{r+p}|Y_1|^s < \infty$ for $r + s \le n + 1$, $s \le n$, $r \le n$, we can make the extra assumption that $EX_1^{n+1}|Y_1|^{n+1} < \infty$. Then if we can find the values of $\delta_1, \cdots, \delta_{n+1}$

in this restricted case, these will also be the coefficients' values in the more general case.

Assuming that $EX_1^{n+1}|Y_1|^{n+1} < \infty$, we can say that $\delta_1, \cdots, \delta_{n+1}$ are rational functions of $\tilde{\mu}_{rs}$ for r and s in $\{0,1,\cdots,n+1\}$. As rational functions of a point in $[(n+2)^2-1]$-dimensional Euclidean space, the δ's are determined uniquely by their values in any sphere in this space. For every positive integer n, we shall find a sphere in which $\delta_1,\cdots,\delta_{n-1}$ vanish identically. This will tell us they are also 0 outside the sphere and will allow us to conclude the nth conditional cumulant is

$$k_{nc}(t) = \delta_n t + \delta_{n+1} + o(t^{-p})$$

whenever the conditions of Theorem 6.1 are fulfilled. We will also find the functional forms of δ_n and δ_{n+1}.

Our arguments are based on the properties of $M_{\theta c}(t)$. We know

$$M_{\theta c}(t) = \frac{\tilde{E}e^{-\sigma\zeta_t + i\theta W(t)}}{\tilde{E}e^{-\sigma\zeta_t}} \equiv \frac{N_\theta(t)}{\tilde{E}e^{-\sigma\zeta_t}}$$

and we consider

$$N_\theta^0(s) = \frac{\omega-\tilde{F}^*(s)}{(s-\sigma)[1-\tilde{G}^*(s,\theta)]} \tag{6.2.6}$$

by equation (6.1.1).

Set $m = (n+2)^2 - 1$. In his 1979 technical report on the cumulants of proper cumulative processes, Smith discussed sequences $\{(X_j,Y_j)\}$ "generated by a λ-model." For such a model, the proper

distribution $G(x,y)$ of (X_1,Y_1) depends solely on the positive parameters $\lambda_1,\cdots,\lambda_m$; also, moments and product moments of all orders exist. He showed that there is a sphere in the m-dimensional space of vectors having coordinates $(\mu_{10},\cdots,\mu_{n+1,0},\cdots, \mu_{n+1,n+1})$ such that these product moments are generated by a λ-model in which $\lambda_1 > \lambda_2>\cdots>\lambda_m$. In these circumstances, he proved that the equation

$$G^*(z,\theta) = 1 \tag{6.2.7}$$

has m distinct roots $z_1(\theta),\cdots, z_n(\theta)$ which are analytic functions of the complex variable θ in some neighborhood of 0. Smith found $z_1(0) = 0$ and for θ in a sufficiently small neighborhood of 0,

$$R\{z_j(\theta)\} < \lambda_1^{-1} \qquad j = 2,3,\cdots,m . \tag{6.2.8}$$

Thus from Smith's results, we know there is an m-dimensional sphere in the space of vectors $(\tilde{\mu}_{10},\cdots, \tilde{\mu}_{n+1,0},\cdots, \tilde{\mu}_{n+1,n+1})$ such that these moments are generated by an appropriate λ-model having the properties already outlined. Let $\tilde{z}_1(\theta),\cdots, \tilde{z}_n(\theta)$ be the roots of

$$\tilde{G}^*(z,\theta) = 1. \tag{6.2.9}$$

In this case we can expand $N_\theta{}^0(z)$ in partial fractions when θ is in a suitable neighborhood of 0;

$$N_\theta{}^0(z) = \frac{\omega-\tilde{F}^*(z)}{(z-\sigma)[1-\tilde{G}^*(z,\theta)]}$$

$$= \sum_{j=1}^{m} \frac{Z_j(\theta)}{z-\tilde{z}_j(\theta)} \tag{6.2.10}$$

where

$$Z_j(\theta) = \frac{\omega - \tilde{F}^*(\tilde{z}_j(\theta))}{(\tilde{z}_j(\theta)-\sigma) \prod_{\substack{k=1 \\ k\neq j}}^{n} (\tilde{z}_j(\theta)-\tilde{z}_k(\theta))} . \tag{6.2.11}$$

Note that for θ in a sufficiently small neighborhood of 0, $R\{\tilde{z}_j(\theta)\} < \frac{\sigma}{2}$ for $j = 1,\cdots, m$ and $\tilde{z}_j(\theta) \neq \tilde{z}_k(\theta)$ for $j \neq k$.

We can easily invert $N_\theta^0(z)$ to find

$$N_\theta(t) = \sum_{j=1}^{n} Z_j(\theta) e^{\tilde{z}_j(\theta)t} . \tag{6.2.12}$$

But since $R\{\tilde{z}_j(\theta)\} < -\lambda_1^{-1}$ for $j=2,3,\cdots,m$, we can conclude that

$$N_\theta(t) = Z_1(\theta) e^{\tilde{z}_1(\theta)t} + O(e^{-t\lambda_1^{-1}}) \tag{6.2.13}$$

where the O term can be shown to be uniform for all sufficiently small $|\theta|$.

Therefore

$$K_{\theta c}(t) = \log M_{\theta c}(t) = \log N_\theta(t) - \log \tilde{E} e^{-\sigma\zeta_t}$$

$$= \tilde{z}_1(\theta)t + \log Z_1(\theta) - \log \tilde{E} e^{-\sigma\zeta_t} + O(e^{-t\lambda_1^{-1}}). \tag{6.2.14}$$

This shows that

$$k_{nc}(t) = \tilde{A}_n t + \tilde{B}_n + O(e^{-t\lambda_1^{-1}}) \tag{6.2.15}$$

whenever $\tilde{G}(x,y)$ is based on an appropriate λ-model. The constant $\tilde{A}_n$ is from the expansion

$$\tilde{z}_1(\theta) = \sum_{j=1}^{\infty} \tilde{A}_j \frac{(i\theta)^j}{j!} \tag{6.2.16}$$

while $\tilde{B}_n$ is from

$$\log Z_1(\theta) = \tilde{B}_0 + \sum_{j=1}^{\infty} \tilde{B}_j \frac{(i\theta)^j}{j!}. \qquad (6.2.17)$$

<u>Theorem 6.2</u>

Suppose $\tilde{F} \in C$ and n is a positive integer. If $\tilde{\mu}_{n+p+1} < \infty$, $\tilde{E}|Y_1|^n < \infty$, and $\tilde{E}X_1^{r+p}|Y_1|^s < \infty$ for $r + s \le n + 1$, $r \le n$, $s \le n$, and $p \le 0$, then

$$k_{nc}(t) = \tilde{A}_n t + \tilde{B}_n + o(t^{-p}).$$

$\tilde{A}_n$ and $\tilde{B}_n$ are rational functions of $\tilde{\mu}_1, \cdots, \tilde{\mu}_{n+1}, \tilde{\kappa}_1, \cdots, \tilde{\kappa}_n$, and $\tilde{\mu}_{rs}$ for $r + s \le n + 1$, $s \le n$, and $r \le n$.

Proof:

From equation (6.2.5) we have

$$k_{nc}(t) = \delta_1 t^n + \delta_2 t^{n-1} + \cdots + \delta_{n+1} + o(t^{-p})$$

and we know the δ's are determined by their values in any sphere. Thus it is enough to consider the δ's resulting from a sphere whose points are generated by a λ-model. But in such a sphere, $\delta_1, \cdots, \delta_{n-1}$ vanish identically and the values of δ_n and δ_{n+1} are given by equations (6.2.16) and (6.2.17). This proves the theorem. □

6.3 <u>THE DERIVATION OF $\tilde{A}_n$ and $\tilde{B}_n$</u>

Smith found that if W(t) is a proper Type B cumulative process generated by a λ-model, then $k_n(t)$, the <u>nth</u> cumulant of W(t), is

$$k_n(t) = A_n t + B_n + O(e^{-t\lambda_1^{-1}})$$

where the generating function for the A_n's is precisely (6.2.16). Thus to find $\tilde{A}_n$ we need only refer to Smith (1979) and assume the joint distribution of (X_1, Y_1) is $\tilde{G}(x,y)$. However, Smith assumed $\tilde{E}Y_1 = 0$ when deriving $A_n = \tilde{A}_n$. As we wish to avoid this assumption, we rederive these coefficients.

Since $\tilde{z}_1(\theta)$ is a root of the equation $\tilde{G}^*(z,\theta) = 1$, we have

$$0 = \tilde{E}[e^{i\theta Y - \tilde{z}_1 X} - 1] = \tilde{E} \sum_{n=1}^{\infty} \frac{1}{n!} [i\theta Y - \tilde{z}_1 X]^n$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!} \tilde{E}[i\theta Y - \tilde{z}_1 X]^n \quad . \tag{6.3.1}$$

Using the expansion $\tilde{z}_1(\theta) = \sum_{j=1}^{\infty} \tilde{A}_j \frac{(i\theta)^j}{j!}$, we find

$$0 = (i\theta)\tilde{\kappa}_1 - \tilde{\mu}_1 \sum_{j=1}^{\infty} \tilde{A}_j \frac{(i\theta)^j}{j!} + \frac{\tilde{\kappa}_2}{2} (i\theta)^2 - (i\theta)\tilde{\mu}_{11} \sum_{j=1}^{\infty} \tilde{A}_j \frac{(i\theta)^j}{j!}$$

$$+ \frac{\tilde{\mu}_2}{2} [\tilde{A}_1^2 (i\theta)^2 + \tilde{A}_1\tilde{A}_2 (i\theta)^3 + (\frac{\tilde{A}_1\tilde{A}_3}{3} + \frac{\tilde{A}_2^2}{4})(i\theta)^4 + \cdots]$$

$$+ \tilde{\kappa}_3 \frac{(i\theta)^3}{3!} - \frac{\tilde{\mu}_{12}}{2} (i\theta)^2 \sum_{j=1}^{\infty} \tilde{A}_j \frac{(i\theta)^j}{j!} + \frac{\tilde{\mu}_{21}}{2} (i\theta) [\tilde{A}_1^2 (i\theta)^2 +$$

$$\tilde{A}_1\tilde{A}_2 (i\theta)^3 + \cdots] - \frac{\tilde{\mu}_3}{3!} [\tilde{A}_1^3 (i\theta)^3 + \frac{3\tilde{A}_1^2\tilde{A}_2}{2} (i\theta)^4 + \cdots] + \tilde{\kappa}_4 \frac{(i\theta)^4}{4!}$$

$$- \frac{\tilde{\mu}_{13}}{6} (i\theta)^3 \sum_{j=1}^{\infty} \tilde{A}_j \frac{(i\theta)^j}{j!} + \frac{\tilde{\mu}_{22}}{4} (i\theta)^2 [\tilde{A}_1^2 (i\theta)^2 + \cdots]$$

$$- \frac{\tilde{\mu}_{31}}{6} (i\theta)[A_1{}^3(i\theta)^3 + \cdots] + \frac{\tilde{\mu}_4}{4!} [\tilde{A}_1{}^4(i\theta)^4 + \cdots] \; . \qquad (6.3.2)$$

Since the coefficient of $\frac{(i\theta)^j}{j!}$ is identically 0 in this expression, we conclude

$$\tilde{A}_1 = \frac{\tilde{\kappa}_1}{\tilde{\mu}_1}$$

$$\tilde{A}_2 = \frac{\tilde{\kappa}_2}{\tilde{\mu}_1} - \frac{2\tilde{\mu}_{11}\tilde{\kappa}_1}{\tilde{\mu}_1{}^2} + \frac{\tilde{\mu}_2\tilde{\kappa}_1{}^2}{\tilde{\mu}_1{}^3}$$

$$\tilde{A}_3 = \frac{-3\tilde{\mu}_{11}\tilde{\kappa}_2}{\tilde{\mu}_1{}^2} + \frac{6\tilde{\mu}_{11}{}^2\tilde{\kappa}_1}{\tilde{\mu}_1{}^3} - \frac{9\tilde{\mu}_{11}\tilde{\mu}_2\tilde{\kappa}_1{}^2}{\tilde{\mu}_1{}^4} + \frac{3\tilde{\mu}_2\tilde{\kappa}_1\tilde{\kappa}_2}{\tilde{\mu}_1{}^3} + \frac{3\tilde{\mu}_2{}^2\tilde{\kappa}_1{}^3}{\tilde{\mu}_1{}^5}$$

$$+ \frac{\tilde{\kappa}_3}{\tilde{\mu}_1} - \frac{3\tilde{\mu}_{12}\tilde{\kappa}_1}{\tilde{\mu}_1{}^2} + \frac{3\tilde{\mu}_{21}\tilde{\kappa}_1{}^2}{\tilde{\mu}_1{}^3} - \frac{\tilde{\mu}_3\tilde{\kappa}_1{}^3}{\tilde{\mu}_1{}^4}$$

$$\tilde{A}_4 = \{4\tilde{A}_3(\tilde{\mu}_2 - \tilde{\mu}_{11}) + 3\tilde{A}_2(\tilde{\mu}_2\tilde{A}_2 - 2\tilde{\mu}_{12} + 4\tilde{\mu}_{21} - 2\tilde{\mu}_3)$$

$$+ \tilde{\kappa}_4 - 4\tilde{\mu}_{13} + 6\tilde{\mu}_{22} - 4\tilde{\mu}_{31} + \tilde{\mu}_4\} \, \tilde{\mu}_1{}^{-1} .$$

Values of $\tilde{A}_n$ for $n > 4$ can be calculated by this same method.

To find $\tilde{B}_n$ we must solve the equation

$$\log Z_1(\theta) = \sum_{j=1}^{\infty} \tilde{B}_j \frac{(i\theta)^j}{j!} + \tilde{B}_0 \; .$$

From equation (6.2.10),

$$Z_1(\theta) = \lim_{z\to\tilde{z}_1(\theta)} [z-\tilde{z}_1(\theta)]N_\theta{}^0(z)$$

$$= \lim_{z\to\tilde{z}_1(\theta)} \frac{[\omega-\tilde{F}^*(z)][z-\tilde{z}_1(\theta)]}{(z-\sigma)[1-\tilde{G}^*(z,\theta)]}$$

$$= \frac{\omega-\tilde{F}^*(z_1)}{\tilde{z}_1-\sigma} \lim_{z\to\tilde{z}_1} \frac{z-\tilde{z}_1(\theta)}{\tilde{G}^*(\tilde{z}_1,\theta)-\tilde{G}^*(z,\theta)}$$

$$= \frac{\omega-\tilde{F}^*(\tilde{z}_1)}{\tilde{z}_1-\sigma} \left[\frac{1}{-\tilde{K}(\theta)} \right] \tag{6.3.3}$$

where $\tilde{K}(\theta) = \frac{\partial}{\partial s} \tilde{G}^*(s,\theta)\Big|_{s=\tilde{z}_1(\theta)}$ is as in equation (4.13) of Smith (1979).

We expand the two factors in (6.3.3) separately.

$$\frac{\omega-\tilde{F}^*(\tilde{z}_1)}{\tilde{z}_1-\sigma} = \frac{\tilde{F}^*(\tilde{z}_1)-\omega}{1-\omega} \cdot \frac{\sigma}{\sigma-z_1} \cdot \frac{1-\omega}{\sigma} \cdot \tag{6.3.4}$$

As the generating function for the $\tilde{B}_n$'s is based on a λ-model, the distribution function $\tilde{F}$ has moments of all orders. We can write

$$\tilde{F}^*(\tilde{z}_1) = 1-\tilde{\mu}_1\tilde{z}_1 + \tilde{\mu}_2 \frac{\tilde{z}_1{}^2}{2!} - \tilde{\mu}_3 \frac{\tilde{z}_1{}^3}{3!} + \tilde{\mu}_4 \frac{\tilde{z}_1{}^4}{4!} - \cdots \tag{6.3.5}$$

$$= 1 - \tilde{\mu}_1 \sum_{j=1}^{\infty} \tilde{A}_j \frac{(i\theta)^j}{j!} + \frac{\tilde{\mu}_2}{2}[\tilde{A}_1{}^2(i\theta)^2+\tilde{A}_1\tilde{A}_3(i\theta)^3+(\frac{\tilde{A}_1\tilde{A}_3}{3} + \frac{\tilde{A}_2{}^2}{4})(i\theta)^4$$

$$+ \cdots] - \frac{\tilde{\mu}_3}{3!}[\tilde{A}_1{}^3(i\theta)^3 + \frac{3\tilde{A}_1{}^2\tilde{A}_2}{2} (i\theta)^4+\cdots] + \frac{\tilde{\mu}_4}{4!}[\tilde{A}_1{}^4(i\theta)^4+\cdots].$$

Therefore

$$\frac{\tilde{F}^*(\tilde{z}_1)-\omega}{1-\omega} = \sum_{j=0}^{\infty} \alpha_j \frac{(i\theta)^j}{j!}$$

where

$$\alpha_0 = 1$$

$$\alpha_1 = -\frac{\tilde{\mu}_1\tilde{A}_1}{1-\omega}$$

$$\alpha_2 = -\frac{\tilde{\mu}_1\tilde{A}_2}{1-\omega} + \frac{\tilde{\mu}_2\tilde{A}_1^{\ 2}}{1-\omega}$$

$$\alpha_3 = -\frac{\tilde{\mu}_1\tilde{A}_3}{1-\omega} + \frac{3\tilde{\mu}_2\tilde{A}_1\tilde{A}_2}{1-\omega} - \frac{\tilde{\mu}_3\tilde{A}_1^{\ 3}}{1-\omega}$$

$$\alpha_4 = -\frac{\tilde{\mu}_1\tilde{A}_4}{1-\omega} + \frac{4\tilde{\mu}_2\tilde{A}_1\tilde{A}_3}{1-\omega} + \frac{3\tilde{\mu}_2\tilde{A}_2^{\ 2}}{1-\omega} - \frac{6\tilde{\mu}_3\tilde{A}_1^{\ 2}\tilde{A}_2}{1-\omega} + \frac{\tilde{\mu}_4\tilde{A}_1^{\ 4}}{1-\omega} .$$

It follows that

$$\log \left[\frac{\tilde{F}^*(\tilde{z}_1)-\omega}{1-\omega}\right] = \sum_{j=1}^{\infty} \alpha_j{}' \frac{(i\theta)^j}{j!} \tag{6.3.6}$$

and

$$\alpha_1{}' = -\frac{\tilde{\mu}_1\tilde{A}_1}{1-\omega}$$

$$\alpha_2{}' = -\frac{\tilde{\mu}_1\tilde{A}_2}{1-\omega} + \frac{\tilde{\mu}_2\tilde{A}_1^{\ 2}}{1-\omega} - \frac{\tilde{\mu}_1^{\ 2}\tilde{A}_1^{\ 2}}{(1-\omega)^2}$$

$$\alpha_3{}' = -\frac{\tilde{\mu}_1\tilde{A}_3}{1-\omega} + \frac{3\tilde{\mu}_2\tilde{A}_1\tilde{A}_2}{1-\omega} - \frac{\tilde{\mu}_3\tilde{A}_1^{\ 3}}{1-\omega} - \frac{3\tilde{\mu}_1^{\ 2}\tilde{A}_1\tilde{A}_2}{(1-\omega)^2}$$

$$+ \frac{3\tilde{\mu}_1\tilde{\mu}_2\tilde{A}_1^{\ 3}}{(1-\omega)^2} - \frac{2\tilde{\mu}_1^{\ 3}\tilde{A}_1^{\ 3}}{(1-\omega)^3}$$

$$\alpha_4' = -\frac{\tilde{\mu}_1\tilde{A}_4}{1-\omega} + \frac{4\tilde{\mu}_2\tilde{A}_1\tilde{A}_3}{1-\omega} + \frac{3\tilde{\mu}_2{}^2\tilde{A}_1{}^2}{1-\omega} - \frac{6\tilde{\mu}_3\tilde{A}_1{}^2\tilde{A}_2}{1-\omega} + \frac{\tilde{\mu}_4\tilde{A}_1{}^4}{1-\omega}$$

$$- \frac{3\tilde{\mu}_1{}^2\tilde{A}_2{}^2}{(1-\omega)^2} + \frac{18\tilde{\mu}_1\tilde{\mu}_2\tilde{A}_1{}^2\tilde{A}_2}{(1-\omega)^2} - \frac{3\tilde{\mu}_2{}^2\tilde{A}_1{}^4}{(1-\omega)^2} - \frac{4\tilde{\mu}_1{}^2\tilde{A}_1\tilde{A}_3}{(1-\omega)^2}$$

$$- \frac{4\tilde{\mu}_1\tilde{\mu}_3\tilde{A}_1{}^4}{(1-\omega)^2} - \frac{12\tilde{\mu}_1{}^3\tilde{A}_1{}^2\tilde{A}_2}{(1-\omega)^3} + \frac{12\tilde{\mu}_1{}^2\tilde{\mu}_2\tilde{A}_1{}^4}{(1-\omega)^3} - \frac{6\tilde{\mu}_1{}^4\tilde{A}_1{}^4}{(1-\omega)^4} .$$

Now we expand $\frac{\sigma}{\sigma-\tilde{z}_1}$ in powers of $(i\theta)$.

$$\frac{\sigma}{\sigma-\tilde{z}_1} = \sum_{k=0}^{\infty}\left(\frac{\tilde{z}_1}{\sigma}\right)^k = 1 + \frac{1}{\sigma}\sum_{j=1}^{\infty}\tilde{A}_j\frac{(i\theta)^j}{j!} +$$

$$\frac{1}{\sigma^2}[\tilde{A}_1{}^2(i\theta)^2 + \tilde{A}_1\tilde{A}_2(i\theta)^3 + (\frac{\tilde{A}_1\tilde{A}_3}{3} + \frac{\tilde{A}_2{}^2}{4})(i\theta)^4 + \cdots]$$

$$+ \frac{1}{\sigma^3}[\tilde{A}_1{}^3(i\theta)^3 + \frac{3\tilde{A}_1{}^2\tilde{A}_2}{2}(i\theta)^4 + \cdots] + \frac{1}{\sigma^4}[\tilde{A}_1{}^4(i\theta)^4 + \cdots] + \cdots$$

$$= \sum_{j=0}^{\infty}\beta_j\frac{(i\theta)^j}{j!} . \qquad (6.3.7)$$

$$\beta_0 = 1$$

$$\beta_1 = \frac{\tilde{A}_1}{\sigma}$$

$$\beta_2 = \frac{\tilde{A}_2}{\sigma} + \frac{2\tilde{A}_1{}^2}{\sigma^2}$$

$$\beta_3 = \frac{\tilde{A}_3}{\sigma} + \frac{6\tilde{A}_1\tilde{A}_2}{\sigma^2} + \frac{6\tilde{A}_1{}^3}{\sigma^3}$$

$$\beta_4 = \frac{\tilde{A}_4}{\sigma} + \frac{8\tilde{A}_1\tilde{A}_2}{\sigma^2} + \frac{6\tilde{A}_2{}^2}{\sigma^2} + \frac{36\tilde{A}_1{}^2\tilde{A}_2}{\sigma^3} + \frac{24\tilde{A}_1{}^4}{\sigma^4} .$$

Since $\frac{\sigma}{\sigma-\tilde{z}_1} = 1 + \sum_{j=1}^{\infty} \beta_j \frac{(i\theta)^j}{j!}$,

$$\log \frac{\sigma}{\sigma-\tilde{z}_1} = \sum_{j=1}^{\infty} \beta_j' \frac{(i\theta)^j}{j!} \tag{6.3.8}$$

where

$$\beta_1' = \frac{\tilde{A}_1}{\sigma}$$

$$\beta_2' = \frac{\tilde{A}_2}{\sigma} + \frac{\tilde{A}_1{}^2}{\sigma^2}$$

$$\beta_3' = \frac{\tilde{A}_3}{\sigma} + \frac{3\tilde{A}_1\tilde{A}_2}{\sigma^2} + \frac{2\tilde{A}_1{}^3}{\sigma^3}$$

$$\beta_4' = \frac{\tilde{A}_4}{\sigma} + \frac{4\tilde{A}_1\tilde{A}_3}{\sigma^2} + \frac{3\tilde{A}_2{}^2}{\sigma^2} + \frac{12\tilde{A}_1{}^2\tilde{A}_2}{\sigma^3} + \frac{6\tilde{A}_1{}^4}{\sigma^4} .$$

Combining results, we find

$$\log \left(\frac{\omega-\tilde{F}^*(\tilde{z}_1)}{\tilde{z}_1-\sigma}\right) = \log\left(\frac{\tilde{F}^*(\tilde{z}_1)-\omega}{1-\omega}\right) + \log\left(\frac{\sigma}{\sigma-\tilde{z}_1}\right) + \log\left(\frac{1-\omega}{\sigma}\right)$$

$$= \sum_{j=1}^{\infty} C_j' \frac{(i\theta)^j}{j!} + \log(\frac{1-\omega}{\sigma}) \tag{6.3.9}$$

where $C_j' = \alpha_j' + \beta_j'$ by equations (6.3.6) and (6.3.8).

We must also expand $-\tilde{K}(\theta)$ in powers of $i\theta$. From Smith (1979), p. 40, we have

$$-\tilde{K}(\theta) = \sum_{j=1}^{\infty} \frac{[-\tilde{z}_1(\theta)]^{j-1}}{(j-1)!} \tilde{\gamma}_j(\theta) \tag{6.3.10}$$

where

$$\tilde{\gamma}_j(\theta) = \sum_{n=0}^{\infty} \frac{\tilde{\mu}_{jn}}{n!} (i\theta)^n. \tag{6.3.11}$$

Thus

$$\frac{-\tilde{K}(\theta)}{\tilde{\mu}_1} = 1 + \frac{1}{\tilde{\mu}_1}\{\tilde{\mu}_{11}(i\theta) + \frac{\tilde{\mu}_{12}}{2}(i\theta)^2 + \frac{\tilde{\mu}_{13}}{3!}(i\theta)^3 + \frac{\tilde{\mu}_{14}}{4!}(i\theta^4)+\cdots\}$$

$$- \frac{1}{\tilde{\mu}_1}\{[\tilde{\mu}_{20}+\tilde{\mu}_{21}(i\theta)+ \frac{\tilde{\mu}_{22}}{2}(i\theta)^2 + \frac{\tilde{\mu}_{23}}{3!}(i\theta)^3+\cdots][\tilde{A}_1(i\theta) + \tilde{A}_2\frac{(i\theta)^2}{2}$$

$$\tilde{A}_3 \frac{(i\theta)^3}{3!} + \tilde{A}_4 \frac{(i\theta)^4}{4!} +\cdots]\}+ \frac{1}{2\tilde{\mu}_1}\{[\tilde{\mu}_{30}+\tilde{\mu}_{31}(i\theta) + \frac{\tilde{\mu}_{32}}{2}(i\theta)^2+\cdots]$$

$$[\tilde{A}_1{}^2(i\theta)^2 + \tilde{A}_1\tilde{A}_2(i\theta)^3+(\frac{\tilde{A}_1\tilde{A}_3}{3} + \frac{\tilde{A}_2{}^2}{4})(i\theta)^4+\cdots]\}$$

$$- \frac{1}{6\tilde{\mu}_1}\{[\tilde{\mu}_{40}+\tilde{\mu}_{41}(i\theta)+\cdots][\tilde{A}_1{}^3(i\theta)^3 + \frac{3\tilde{A}_1{}^2\tilde{A}_2}{2}(i\theta)^4+\cdots]\}$$

$$+ \frac{1}{24\tilde{\mu}_1}\{[\tilde{\mu}_{50}+\cdots][\tilde{A}_1{}^4(i\theta)^4+\cdots]\} + \cdots$$

$$= \sum_{j=0}^{\infty} k_j \frac{(i\theta)^j}{j!} \qquad (6.3.12)$$

where

$$k_0 = 1$$

$$k_1 = \frac{1}{\tilde{\mu}_1}[\tilde{\mu}_{11}-\tilde{\mu}_2\tilde{A}_1]$$

$$k_2 = \frac{1}{\tilde{\mu}_1}[\tilde{\mu}_{12}-\tilde{\mu}_2\tilde{A}_2 - 2\tilde{\mu}_{21}\tilde{A}_1 + \tilde{\mu}_3\tilde{A}_1{}^2]$$

$$k_3 = \frac{1}{\tilde{\mu}_1}[\tilde{\mu}_{13}-\tilde{\mu}_2\tilde{A}_3 - 3\tilde{\mu}_{21}\tilde{A}_2 - 3\tilde{\mu}_{22}\tilde{A}_1 + 3\tilde{\mu}_3\tilde{A}_1\tilde{A}_2 + 3\tilde{\mu}_{31}\tilde{A}_1{}^2 - \tilde{\mu}_4\tilde{A}_1{}^3]$$

$$k_4 = \frac{1}{\tilde{\mu}_1}[\tilde{\mu}_{14}-\tilde{\mu}_2\tilde{A}_4 - 4\tilde{\mu}_{21}\tilde{A}_3 - 6\tilde{\mu}_{22}\tilde{A}_2 - 4\tilde{\mu}_{23}\tilde{A}_1 + 4\tilde{\mu}_3\tilde{A}_1\tilde{A}_3$$

$$+ 3\tilde{\mu}_3\tilde{A}_2{}^2 + 12\tilde{\mu}_{31}\tilde{A}_1\tilde{A}_2 + 6\tilde{\mu}_{32}\tilde{A}_1{}^2 - 6\tilde{\mu}_4\tilde{A}_1{}^2\tilde{A}_2 - 4\tilde{\mu}_{41}\tilde{A}_1{}^3 + \tilde{\mu}_5\tilde{A}_1{}^4].$$

Therefore

$$\log[-\frac{\tilde{K}(\theta)}{\tilde{\mu}_1}] = \sum_{j=1}^{\infty} k_j' \frac{(i\theta)^j}{j!}$$

where

$$k_1' = \frac{1}{\tilde{\mu}_1}[\tilde{\mu}_{11}-\tilde{\mu}_2\tilde{A}_1]$$

$$k_2' = \frac{1}{\tilde{\mu}_1}[\tilde{\mu}_{12}-\tilde{\mu}_2\tilde{A}_2 - 2\tilde{\mu}_{21}\tilde{A}_1 + \tilde{\mu}_3\tilde{A}_1^{\,2}] - \frac{1}{\tilde{\mu}_1^{\,2}}[\tilde{\mu}_{11}^{\,2} - 2\tilde{\mu}_{11}\tilde{\mu}_2\tilde{A}_1 + \tilde{\mu}_2^{\,2}\tilde{A}_1^{\,2}]$$

$$k_3' = \frac{1}{\tilde{\mu}_1}[\tilde{\mu}_{13}-\tilde{\mu}_2\tilde{A}_3 - 3\tilde{\mu}_{21}\tilde{A}_2 - 3\tilde{\mu}_{22}\tilde{A}_1 + 3\tilde{\mu}_3\tilde{A}_1\tilde{A}_2 + 3\tilde{\mu}_{31}\tilde{A}_1^{\,2} - \tilde{\mu}_4\tilde{A}_1^{\,3}]$$

$$- \frac{3}{\tilde{\mu}_1^{\,2}}[\tilde{\mu}_{11}\tilde{\mu}_{12} - \tilde{\mu}_{11}\tilde{\mu}_2\tilde{A}_2 - 2\tilde{\mu}_{11}\tilde{\mu}_{21}\tilde{A}_1 + \tilde{\mu}_{11}\tilde{\mu}_3\tilde{A}_1^{\,2} - \tilde{\mu}_{12}\tilde{\mu}_2\tilde{A}_1$$

$$+ \tilde{\mu}_2^{\,2}\tilde{A}_1\tilde{A}_2 + 2\tilde{\mu}_2\tilde{\mu}_{21}\tilde{A}_1^{\,2} - \tilde{\mu}_2\tilde{\mu}_3\tilde{A}_1^{\,3}]$$

$$+ \frac{2}{\tilde{\mu}_1^{\,3}}[\tilde{\mu}_{11}^{\,3} - 3\tilde{\mu}_{11}^{\,2}\tilde{\mu}_2\tilde{A}_1 + 3\tilde{\mu}_{11}\tilde{A}_1^{\,2} - \tilde{\mu}_2^{\,3}\tilde{A}_1^{\,3}]$$

$$k_4' = 4k_4 - 3k_2^{\,2} - 4k_1k_3 + 12k_1^{\,2}k_2 - 6k_1^{\,4}.$$

Returning to equation (6.3.3), we have

$$\log Z_1(\theta) = \log\left(\frac{\omega-\tilde{F}^*(z_1)}{\tilde{z}_1-\sigma}\right) - \log(-\tilde{K}(\theta))$$

$$= \sum_{j=1}^{\infty} C_j' \frac{(i\theta)^j}{j!} + \log\left(\frac{1-\omega}{\sigma}\right) - \log\left(\frac{-\tilde{K}(\theta)}{\tilde{\mu}_1}\right) - \log(\tilde{\mu}_1)$$

$$= \sum_{j=1}^{\infty} \tilde{B}_j \frac{(i\theta)^j}{j!} + \log\left(\frac{1-\omega}{\sigma\tilde{\mu}_1}\right) \tag{6.3.13}$$

and

$$\tilde{B}_j = C_j' - k_j'$$

$$\tilde{B}_1 = -\frac{\tilde{\mu}_1\tilde{A}_1}{1-\omega} + \frac{\tilde{A}_1}{\sigma} - \frac{\tilde{\mu}_{11}}{\tilde{\mu}_1} + \frac{\tilde{\mu}_2\tilde{A}_1}{\tilde{\mu}_1}$$

$$\tilde{B}_2 = -\frac{\tilde{\mu}_1\tilde{A}_2}{1-\omega} + \frac{\tilde{\mu}_2\tilde{A}_1^2}{1-\omega} - \frac{\tilde{\mu}_1^2\tilde{A}_1^2}{(1-\omega)^2} + \frac{\tilde{A}_2}{\sigma} + \frac{\tilde{A}_1^2}{\sigma^2}$$

$$-\frac{1}{\tilde{\mu}_1}[\tilde{\mu}_{12}-\tilde{\mu}_2\tilde{A}_2 - 2\tilde{\mu}_{21}\tilde{A}_1 + \tilde{\mu}_3\tilde{A}_1^2] + \frac{1}{\tilde{\mu}_1^2}[\tilde{\mu}_{11}^2 - 2\tilde{\mu}_{11}\tilde{\mu}_2\tilde{A}_1 + \tilde{\mu}_2^2\tilde{A}_1^2]$$

$$\tilde{B}_3 = -\frac{\tilde{\mu}_1\tilde{A}_3}{1-\omega} + \frac{3\tilde{\mu}_2\tilde{A}_1\tilde{A}_2}{1-\omega} - \frac{\tilde{\mu}_3\tilde{A}_1^3}{1-\omega} - \frac{3\tilde{\mu}_1^2\tilde{A}_1\tilde{A}_2}{(1-\omega)^2} + \frac{3\tilde{\mu}_1\tilde{\mu}_2\tilde{A}_1^3}{(1-\omega)^2}$$

$$-\frac{2\tilde{\mu}_1^3\tilde{A}_1^3}{(1-\omega)^3} + \frac{\tilde{A}_3}{\sigma} + \frac{3\tilde{A}_1\tilde{A}_2}{\sigma^2} + \frac{2\tilde{A}_1^3}{\sigma^3}$$

$$-\frac{1}{\tilde{\mu}_1}[\tilde{\mu}_{13}-\tilde{\mu}_2\tilde{A}_3 - 3\tilde{\mu}_{21}\tilde{A}_2 - 3\tilde{\mu}_{22}\tilde{A}_1 + 3\tilde{\mu}_3\tilde{A}_1\tilde{A}_2 + 3\tilde{\mu}_{31}\tilde{A}_1^2 - \tilde{\mu}_4\tilde{A}_1^3]$$

$$+\frac{3}{\tilde{\mu}_1^2}[\tilde{\mu}_{11}\tilde{\mu}_{12}-\tilde{\mu}_{11}\tilde{\mu}_2\tilde{A}_2 - 2\tilde{\mu}_{11}\tilde{\mu}_{21}\tilde{A}_1 + \tilde{\mu}_{11}\tilde{\mu}_3\tilde{A}_1^2 - \tilde{\mu}_{12}\tilde{\mu}_2\tilde{A}_1 + \tilde{\mu}_2^2\tilde{A}_1\tilde{A}_2$$

$$+ 2\tilde{\mu}_2\tilde{\mu}_{21}\tilde{A}_1^2 - \tilde{\mu}_2\tilde{\mu}_3\tilde{A}_1^3] - \frac{2}{\tilde{\mu}_1^3}[\tilde{\mu}_{11}^3 - 3\tilde{\mu}_{11}^2\tilde{\mu}_2\tilde{A}_1 + 3\tilde{\mu}_{11}\tilde{\mu}_2^2\tilde{A}_1^2 - \tilde{\mu}_2^3\tilde{A}_1^3].$$

Values of $\tilde{B}_n$ for $n \geq 4$ can be derived by careful arithmetic.

As a check on our derivation of $\tilde{A}_n$ and $\tilde{B}_n$, we compute their values for the special case $Y_j \equiv 1$. The formula

$$k_{nc}(t) = \tilde{A}_n t + \tilde{B}_n + o(t^{-p})$$

should then reduce to the formula for the nth conditional cumulant of N(t) found in Chapter 5.

When $Y_j \equiv 1$, $\tilde{z}^s = 1$ for all s and $\tilde{\mu}_{rs} \equiv \tilde{\mu}_r$. The first three $\tilde{A}_n$'s reduce to

$$\tilde{A}_1 = \frac{1}{\tilde{\mu}_1}$$

$$\tilde{A}_2 = \frac{\tilde{\mu}_2}{\tilde{\mu}_1^{\,3}} - \frac{1}{\tilde{\mu}_1}$$

$$\tilde{A}_3 = \frac{3\tilde{\mu}_2^{\,2}}{\tilde{\mu}_1^{\,5}} - \frac{\tilde{\mu}_3}{\tilde{\mu}_1^{\,4}} - \frac{3\tilde{\mu}_2}{\tilde{\mu}_1^{\,3}} + \frac{1}{\tilde{\mu}_1}$$

and the first three $\tilde{B}_n$'s are

$$\tilde{B}_1 = \frac{\tilde{\mu}_2}{\tilde{\mu}_1^{\,2}} + \frac{1}{\sigma\tilde{\mu}_1} - \frac{2-\omega}{1-\omega}$$

$$\tilde{B}_2 = \frac{2\tilde{\mu}_2^{\,2}}{\tilde{\mu}_1^{\,4}} - \frac{\tilde{\mu}_3}{\tilde{\mu}_1^{\,3}} - \frac{\tilde{\mu}_2}{\tilde{\mu}_1^{\,2}} + \frac{1}{\sigma^2\tilde{\mu}_1^{\,2}} - \frac{1}{\sigma\tilde{\mu}_1} + \frac{\tilde{\mu}_2}{\sigma\tilde{\mu}_1^{\,3}} - \frac{\omega}{(1-\omega)^2}$$

$$\tilde{B}_3 = \frac{8\tilde{\mu}_2^{\,3}}{\tilde{\mu}_1^{\,6}} - \frac{7\tilde{\mu}_2\tilde{\mu}_3}{\tilde{\mu}_1^{\,5}} + \frac{\tilde{\mu}_4}{\tilde{\mu}_1^{\,4}} + \frac{3\tilde{\mu}_2^{\,2}}{\sigma\tilde{\mu}_1^{\,5}} - \frac{6\tilde{\mu}_2^{\,2}}{\tilde{\mu}_1^{\,4}} - \frac{\tilde{\mu}_3}{\sigma\tilde{\mu}_1^{\,4}}$$

$$+ \frac{3\tilde{\mu}_3}{\tilde{\mu}_1^{\,3}} + \frac{3\tilde{\mu}_2}{\sigma^2\tilde{\mu}_1^{\,4}} - \frac{3\tilde{\mu}_2}{\sigma\tilde{\mu}_1^{\,3}} + \frac{\tilde{\mu}_2}{\tilde{\mu}_1^{\,2}} + \frac{2}{\sigma^3\tilde{\mu}_1^{\,3}} - \frac{3}{\sigma^2\tilde{\mu}_1^{\,2}}$$

$$+ \frac{1}{\sigma\tilde{\mu}_1} - \frac{2}{(1-\omega)^3} + \frac{3}{(1-\omega)^2} - \frac{1}{1-\omega} .$$

Thus we have that if $\tilde{\mu}_{n+p+1} < \infty$ and $\tilde{F} \in C$

a) and $n = 1$, then

$$k_{1c}(t) = \frac{t}{\tilde{\mu}_1} + \frac{\tilde{\mu}_2}{\tilde{\mu}_1^{\,2}} + \frac{1}{\sigma\tilde{\mu}_1} - \frac{2-\omega}{1-\omega} + o(t^{-p})$$

b) and $n = 2$, then

$$k_{2c}(t) = \left(\frac{\tilde{\mu}_2}{\tilde{\mu}_1^{\,3}} - \frac{1}{\tilde{\mu}_1}\right)t + \frac{2\tilde{\mu}_2^{\,2}}{\tilde{\mu}_1^{\,4}} - \frac{\tilde{\mu}_3}{\tilde{\mu}_1^{\,3}} - \frac{\tilde{\mu}_2}{\tilde{\mu}_1^{\,2}} + \frac{1}{\sigma^2\tilde{\mu}_1^{\,2}}$$

$$- \frac{1}{\sigma\tilde{\mu}_1} + \frac{\tilde{\mu}_2}{\sigma\tilde{\mu}_1^{\,3}} - \frac{\omega}{(1-\omega)^2} + o(t^{-p})$$

c) and $n = 3$, then

$$k_{3c}(t) = \left(\frac{3\tilde{\mu}_2^{\,2}}{\tilde{\mu}_1^{\,5}} - \frac{\tilde{\mu}_3}{\tilde{\mu}_1^{\,4}} - \frac{3\tilde{\mu}_2}{\tilde{\mu}_1^{\,3}} + \frac{1}{\tilde{\mu}_1}\right)t + \frac{8\tilde{\mu}_2^{\,3}}{\tilde{\mu}_1^{\,6}} - \frac{7\tilde{\mu}_2\tilde{\mu}_3}{\tilde{\mu}_1^{\,5}}$$

$$+ \frac{\tilde{\mu}_4}{\tilde{\mu}_1^{\,4}} + \frac{3\tilde{\mu}_2^{\,2}}{\sigma\tilde{\mu}_1^{\,5}} - \frac{6\tilde{\mu}_2^{\,2}}{\tilde{\mu}_1^{\,4}} - \frac{\tilde{\mu}_3}{\sigma\tilde{\mu}_1^{\,4}} + \frac{3\tilde{\mu}_3}{\tilde{\mu}_1^{\,3}} + \frac{3\tilde{\mu}_2}{\sigma^2\tilde{\mu}_1^{\,4}} - \frac{3\tilde{\mu}_2}{\sigma\tilde{\mu}^3} + \frac{\tilde{\mu}_2}{\tilde{\mu}_1^{\,2}}$$

$$+ \frac{2}{\sigma^3\tilde{\mu}_1^{\,3}} - \frac{3}{\sigma^2\tilde{\mu}_1^{\,2}} + \frac{1}{\sigma\tilde{\mu}_1} - \frac{2}{(1-\omega)^3} + \frac{3}{(1-\omega)^2} - \frac{1}{1-\omega} + o(t^{-p}).$$

Since these formulas are the same as those derived in Chapter 5, we believe the cited values of $\tilde{A}_n$ and $\tilde{B}_n$ are correct.

Finally, we return to a question raised in Chapter 4. We assumed $\tilde{\mu}_2 < \infty$ and $\tilde{\kappa}_2^{*} < \infty$ and asked whether

$$\mathrm{Var}[W(t)\,|\,A(t) = 1] = \frac{\gamma t}{\tilde{\mu}_1} + o(t).$$

We concluded that the statement is true if

$$E[W(t)|A(t)=1] - \frac{\tilde{\kappa}_1 t}{\tilde{\mu}_1} = o(\sqrt{t}).$$

The assumption that $\tilde{\mu}_2 < \infty$ and $\tilde{\kappa}_2^* < \infty$ implies $\tilde{E}X_1|Y_1| < \infty$; if we also suppose that $\tilde{F} \in C$ and $W(t)$ is a Type B process, Theorem 6.2 yields

$$E[W(t)|A(t)=1] = \tilde{A}_1 t + \tilde{B}_1 + o(1).$$

Since $\tilde{A}_1 = \frac{\tilde{\kappa}_1}{\tilde{\mu}_1}$, the result is true for these special cumulative processes.

Of course, if we are willing to assume more about the moments and product moments of (X_1,Y_1), we can specify the error term much more precisely. It is worthwhile to note that $\tilde{A}_2$, the coefficient of t in $Var[W(t)|A(t)=1]$ under the conditions of Theorem 6.2, is $\frac{\tilde{\gamma}}{\tilde{\mu}_1}$, as we know it should be. This is a further check on our calculations.

CHAPTER 7

PROCESSES WITH VARIABLE DEFECTS

Although we have assumed that each lifetime X_i has the same defect $1-\omega$, there may be cases in which it is reasonable to suppose that the defect depends on the number of lifetimes preceding X_i. Let $\{\omega_j\}$ be a sequence in $(0,1]$ and suppose $\{X_i\}$ is a sequence of independent positive random variables such that $P\{X_i \le x\} = \omega_i J(x)$ where J is a proper distribution on $[0,\infty)$ and $J(0^+) < 1$. Note that

$$P\{X_1+X_2 \le t\} = \int_0^t \omega_2 J(t-\tau)\omega_1 dJ(\tau) = \omega_1\omega_2 J^{(2)}(t)$$

and in general

$$P\{S_n \le t\} = \left(\prod_{j=1}^{n} \omega_j\right) J^{(n)}(t). \tag{7.1}$$

It follows immediately that

$$H(t) = \sum_{n=1}^{\infty} \left(\prod_{j=1}^{n} \omega_j\right) J^{(n)}(t) \tag{7.2}$$

and

$$H(\infty) = \sum_{n=1}^{\infty} \left(\prod_{j=1}^{n} \omega_j\right) \le \infty.$$

If the ω_j's are not bounded above by some $\omega < 1$, it is entirely possible that $H(\infty) = \infty$. Also, unlike the situation in which

$\omega_j \le \omega$, the probability the process is alive at time t does not necessarily converge to 0. We find

$$q(t) = P\{A(t) = 1\}$$

$$= \omega_1[1-J(t)] + \sum_{n=1}^{\infty} \int_0^t \omega_{n+1}[1-J(t-\tau)](\prod_{j=1}^{n} \omega_j)dJ^{(n)}(\tau)$$

$$= \sum_{n=0}^{\infty} (\prod_{j=1}^{n+1} \omega_j)[J^{(n)}(t)-J^{(n+1)}(t)]. \tag{7.3}$$

Lemma 7.1

$$q(t) = \omega_1 - \sum_{n=1}^{\infty}(1-\omega_{n+1})(\prod_{j=1}^{n} \omega_j)J^{(n)}(t). \tag{7.4}$$

Proof:

The right hand side of (7.4) equals

$$\omega_1 - \sum_{n=1}^{\infty}(1-\omega_{n+1})(\prod_{j=1}^{n} \omega_j)[J^{(n)}(t)-J^{(n+1)}(t)]$$

$$- \sum_{n=1}^{\infty}(1-\omega_{n+1})(\prod_{j=1}^{n} \omega_j)J^{(n+1)}(t)$$

$$= \omega_1 - \sum_{n=1}^{\infty}(\prod_{j=1}^{n} \omega_j)[J^{(n)}(t)-J^{(n+1)}(t)]$$

$$+ \sum_{n=1}^{\infty}(\prod_{j=1}^{n+1} \omega_j)[J^{(n)}(t)-J^{(n+1)}(t)] - \sum_{n=1}^{\infty}(1-\omega_{n+1})(\prod_{j=1}^{n} \omega_j)J^{(n+1)}(t)$$

$$= \sum_{n=0}^{\infty} \left(\prod_{j=1}^{n+1} \omega_j\right)[J^{(n)}(t)-J^{(n+1)}(t)] + \{\omega_1 J(t)$$

$$- \sum_{n=1}^{\infty} \left(\prod_{j=1}^{n} \omega_j\right)[J^{(n)}(t)-J^{(n+1)}(t)]$$

$$- \sum_{n=1}^{\infty} (1-\omega_{n+1})\left(\prod_{j=1}^{n} \omega_j\right)J^{(n+1)}(t)\}$$

$$= q(t) + \sum_{j=1}^{\infty} \alpha_j J^{(j)}(t). \tag{7.5}$$

Note that $\alpha_1 = \omega_1 - \omega_1 = 0$ and

$$\alpha_k = -\prod_{j=1}^{k} \omega_j + \prod_{j=1}^{k-1} \omega_j - (1-\omega_k)\prod_{j=1}^{k-1} \omega_j = 0. \qquad \square$$

Formula (7.4) shows the monotonicity of q(t) much more clearly than (7.3).

Because $\prod_{j=1}^{n} \omega_j$ is nonincreasing in n, it converges to some limit $\ell \geq 0$. In fact, since we are assuming $\omega_j > 0$ for all j, ℓ is positive if and only if $\sum_{j=1}^{\infty} (1-\omega_j) < \infty$.

<u>Lemma 7.2</u>

$$\lim_{t\to\infty} q(t) = \lim_{n\to\infty} \prod_{j=1}^{n} \omega_j \equiv \ell .$$

Proof:

$$q(t) \geq \ell \sum_{n=0}^{\infty} [J^{(n)}(t)-J^{(n+1)}(t)]$$

$$= \ell[1- \lim_{n\to\infty} J^{(n)}(t)] = \ell. \tag{7.6}$$

Now choose $\varepsilon > 0$. There exists some $N(\varepsilon)$ such that $\prod_{j=1}^{n+1} \omega_j < \ell + \varepsilon$ whenever $n \geq N$. Thus

$$\lim_{t\to\infty} q(t) \leq \lim_{t\to\infty} \sum_{n=0}^{N-1} \left(\prod_{j=1}^{n+1} \omega_j\right)[J^{(n)}(t)-J^{(n+1)}(t)]$$

$$+ \lim_{t\to\infty} (\ell+\varepsilon) \sum_{n=N}^{\infty} [J^{(n)}(t)-J^{(n+1)}(t)]$$

$$\leq \lim_{t\to\infty} [1-J^{(N)}(t)] + \ell + \varepsilon$$

$$= \ell + \varepsilon. \tag{7.7}$$

Since ε is arbitrary, $\lim_{t\to\infty} q(t) = \ell$. □

<u>Lemma 7.3</u>

Suppose $\sum_{n=1}^{\infty} \left(\prod_{j=1}^{n} \omega_j\right) = \infty$. Then

$P\{A(t+s) = 1|A(t)=1\} \to 1$ as $t \to \infty$.

Proof:

If $\prod_{j=1}^{\infty} \omega_j = \ell > 0$, then by Lemma 7.2 $\lim_{t\to\infty} \frac{q(t+s)}{q(t)} = 1$.

Suppose then that $\lim_{n\to\infty} \prod_{j=1}^{n} \omega_j = 0$. We will first show that $P\{N(t)=k|A(t)=1\} \to 0$ for fixed k.

$$P\{N(t)=k|A(t)=1\} = \frac{\left(\prod_{j=1}^{k+1} \omega_j\right)[J^{(k)}(t)-J^{(k+1)}(t)]}{\sum_{n=0}^{\infty}\left(\prod_{j=1}^{n+1} \omega_j\right)[J^{(n)}(t)-J^{(n+1)}(t)]}$$

$$\le \frac{J^{(k)}(t) - J^{(k+1)}(t)}{1-J^{(k+1)}(t)+\sum_{n=k+1}^{\infty}\left(\prod_{j=k+2}^{n+1} \omega_j\right)[J^{(n)}(t)-J^{(n+1)}(t)]}. \quad (7.8)$$

Observe that

$$1 = (1-\omega_{n+2}) + \sum_{m=n+3}^{\infty}(1-\omega_m)\prod_{p=n+2}^{m-1}\omega_p \qquad n = 1, 2, \ldots$$

since $\lim_{n\to\infty}\prod_{j=1}^{n}\omega_j = 0$. Hence

$$\left(\prod_{j=k+2}^{n+1} \omega_j\right)[J^{(n)}(t)-J^{(n+1)}(t)]$$

$$= [J^{(n)}(t)-J^{(n+1)}(t)]\{(1-\omega_{n+2})\left(\prod_{j=k+2}^{n+1}\omega_j\right)+\sum_{m=n+3}^{\infty}(1-\omega_m)\prod_{p=k+2}^{m-1}\omega_p\}$$

$$= [J^{(n)}(t)-J^{(n+1)}(t)]\sum_{m=n+2}^{\infty}(1-\omega_m)\prod_{j=k+2}^{m-1}\omega_j \quad . \qquad (7.9)$$

Thus it follows that

$$\sum_{n=k+1}^{\infty}\left(\prod_{j=k+2}^{n+1} \omega_j\right)[J^{(n)}(t)-J^{(n+1)}(t)]$$

$$= \sum_{n=k+1}^{\infty} [J^{(n)}(t) - J^{(n+1)}(t)] \sum_{m=n+2}^{\infty} (1-\omega_m) \prod_{j=k+2}^{m-1} \omega_j$$

$$= \sum_{m=k+3}^{\infty} (1-\omega_m) \Big(\prod_{j=k+2}^{m-1} \omega_j \Big) \sum_{n=k+1}^{m-2} [J^{(n)}(t) - J^{(n+1)}(t)]$$

$$= \sum_{m=k+3}^{\infty} (1-\omega_m) \Big(\prod_{j=k+2}^{m-1} \omega_j \Big) [J^{(k+1)}(t) - J^{(m-1)}(t)]. \tag{7.10}$$

Also,

$$1-J^{(k+1)}(t) = [1-J^{(k+1)}(t)]\{1-\omega_{k+2} + \sum_{m=k+3}^{\infty} (1-\omega_m) \prod_{j=k+2}^{m-1} \omega_j\}.$$

Therefore,

$$1-J^{(k+1)}(t) + \sum_{n=k+1}^{\infty} \Big(\prod_{j=k+2}^{n+1} \omega_j \Big) [J^{(n)}(t) - J^{(n+1)}(t)]$$

$$(7.11)$$

$$= (1-\omega_{k+2})[1-J^{(k\ 1)}(t)] + \sum_{m=k+3}^{\infty} (1-\omega_m) \Big(\prod_{j=k+2}^{m-1} \omega_j \Big) [1-J^{(m-1)}(t)]$$

and we have

$$P\{N(t)=k \,|\, A(t)=1\} \tag{7.12}$$

$$\leq \frac{1-J^{(k+1)}(t)}{(1-\omega_{k+2})[1-J^{(k+1)}(t)] + \sum_{m=k+2}^{\infty} (1-\omega_{m+1}) \Big(\prod_{j=k+2}^{m} \omega_j \Big) [1-J^{(m)}(t)]}.$$

Now define $\alpha_m = [\frac{m}{k+1}]$.

$$J^{(m)}(t) \le J^{(\alpha_m(k+1))}(k) \le [J^{(k+1)}(t)]^{\alpha_m}$$

and thus

$$1-J^{(m)}(t) \ge 1-[J^{(k+1)}(t)]^{\alpha_m} . \tag{7.13}$$

Combining (7.12) and (7.13), we see

$$P\{N(t)=k|A(t)=1\}$$

$$\le \frac{1 - J^{(k+1)}(t)}{(1-\omega_{k+2})[1-J^{(k+1)}(t)]+\sum_{m=k+2}^{\infty}(1-\omega_{m+1})(\prod_{j=k+2}^{m}\omega_j)\{1-[J^{(k+1)}(t)]^{\alpha_m}\}}$$

$$= \frac{1}{(1-\omega_{k+2})+\sum_{m=k+2}^{\infty}(1-\omega_{m+1})(\prod_{j=k+2}^{m}\omega_j)\{1+J^{(k+1)}(t)+\cdots+[J^{(k+1)}(t)]^{\alpha_m-1}\}} .$$

(7.14)

Note that

$$\sum_{m=k+2}^{\infty}(1-\omega_{m+1})(\prod_{j=k+2}^{m}\omega_j)\alpha_m > c\sum_{m=k+2}^{\infty}m(1-\omega_{m+1})(\prod_{j=k+2}^{m}\omega_j)$$

for some $c > 0$. But

$$\sum_{m=k+2}^{\infty} m(1-\omega_{m+1})\left(\prod_{j=k+2}^{m} \omega_j\right) = (k+1) \sum_{m=k+2}^{\infty} (1-\omega_{m+1})\left(\prod_{j=k+2}^{m} \omega_j\right)$$

$$+ \sum_{m=k+2}^{\infty} \sum_{\ell=1}^{m-k-1} (1-\omega_{m+1})\left(\prod_{j=k+2}^{m} \omega_j\right)$$

$$= (k+1) \sum_{m=k+2}^{\infty} (1-\omega_{m+1})\left(\prod_{j=k+2}^{m} \omega_j\right) + \sum_{\ell=k+2}^{\infty} \sum_{m=\ell}^{\infty} (1-\omega_{m+1}) \prod_{j=k+2}^{m} \omega_j \ . \tag{7.15}$$

Now since

$$\prod_{j=1}^{n} \omega_j \to 0, \quad \sum_{m=\ell}^{\infty} (1-\omega_{m+1}) \prod_{j=k+2}^{m} \omega_j = \prod_{j=k+2}^{\ell} \omega_j$$

and so

$$\sum_{m=k+2}^{\infty} m(1-\omega_{m+1})\left(\prod_{j=k+2}^{m} \omega_j\right) = (k+1) \sum_{m=k+2}^{\infty} (1-\omega_{m+1})\left(\prod_{j=k+2}^{m} \omega_j\right)$$

$$+ \sum_{\ell=k+2}^{\infty} \left(\prod_{j=k+2}^{\ell} \omega_j\right) = \infty \tag{7.16}$$

by assumption. Therefore

$$\lim_{t\to\infty} P\{N(t)=k \mid A(t)=1\} = 0.$$

Now we use this fact to show

$$\lim_{t\to\infty} \frac{q(t+s)}{q(t)} = 1 \quad \text{for fixed } s.$$

Choose $\varepsilon > 0$ and fix s. There exists some $N(\varepsilon,s)$ such that for all $n \geq N$, $1- J^{(n)}(s)>1-\varepsilon$. Also, since $\sum_{n=1}^{\infty} (\prod_{j=1}^{n} \omega_j) = \infty$, $\omega_j \to 1$ and there is some $K(N)$ making

$$\prod_{j=k+1}^{k+1+N} \omega_j > 1-\varepsilon \quad \text{whenever } k \geq K.$$

$$\frac{q(t+s)}{q(t)} = \frac{P\{A(t+s)=1,N(t)<K\}}{P\{A(t)=1\}} + \frac{P\{A(t+s)=1,N(t)\geq K\}}{P\{A(t)=1\}}$$

$$\geq \frac{P\{A(t+s)=1,N(t)\geq K\}}{P\{N(t)\geq K,A(t)=1\}} \quad \frac{P\{N(t)\geq K,A(t)=1\}}{P\{A(t)=1\}} . \tag{7.17}$$

We already know

$$\frac{P\{N(t)\geq K,A(t)=1\}}{P\{A(t)=1\}} \to 1$$

and now consider

$$P\{A(t+s)=1,N(t)\geq K\} = \sum_{n=K}^{\infty} P\{A(t+s)=1,N(t)=n\}$$

$$= \sum_{n=K}^{\infty} P\{A(t+s)=1 \mid N(t)=n,A(t)=1\}P\{N(t)=n,A(t)=1\}$$

$$\geq \sum_{n=K}^{\infty} P\{N(t)=n,A(t)=1\}P\{A(t+\zeta_t+s)=1 \mid N(t)=n,A(t)=1\}$$

$$= \sum_{n=K}^{\infty} P\{N(t)=n, A(t)=1\} \sum_{m=0}^{\infty} \left(\prod_{j=n+1}^{n+m+1} \omega_j\right)[J^{(m)}(s) - J^{(m+1)}(s)]$$

$$\geq \sum_{n=K}^{\infty} P\{N(t)=n, A(t)=1\} \sum_{m=0}^{N} \left(\prod_{j=n+1}^{n+1+N} \omega_j\right)[J^{(m)}(s) - J^{(m+1)}(s)]$$

$$> (1-\varepsilon) \sum_{n=K}^{\infty} P\{N(t)=n, A(t)=1\}[1 - J^{(N+1)}(s)]$$

$$> (1-\varepsilon)^2 P\{N(t) \geq K, A(t)=1\}. \tag{7.18}$$

Hence

$$\lim_{t\to\infty} \frac{q(t+s)}{q(t)} \geq (1-\varepsilon)^2.$$

Since ε is arbitrary this proves the result. □

In Chapter 3 we learned that if $J(x) \in \mathcal{S}$, the class of subexponential distributions, and if $P\{X_i \leq x\} = \omega J(x)$, then $\lim_{t\to\infty} \frac{q(t+s)}{q(t)} = 1$. Our next lemma generalizes this result.

<u>Lemma 7.4</u>

Suppose $\omega_n \leq \omega < 1$ for all n and $J(x) \in \mathcal{S}$. Then

$$\lim_{t\to\infty} \frac{q(t+s)}{q(t)} = 1.$$

Proof:

Since

$$\omega_n \le \omega, \; 1 = 1 - \omega_1 + \sum_{n=1}^{\infty} (1-\omega_{n+1})(\prod_{j=1}^{n} \omega_j)$$

or

$$\omega_1 = \sum_{n=1}^{\infty} (1-\omega_{n+1})(\prod_{j=1}^{n} \omega_j) \; . \tag{7.19}$$

Combining (7.19) and expression (7.4) for q(t), we discover

$$q(t) = \sum_{n=1}^{\infty} (1-\omega_{n+1})(\prod_{j=1}^{n} \omega_j)[1-J^{(n)}(t)]. \tag{7.20}$$

Hence

$$\lim_{t\to\infty} \frac{q(t+s)}{q(t)} = \left\{\frac{1-J(t+s)}{1-J(t)}\right\} \left\{\frac{\sum_{n=1}^{\infty} (1-\omega_{n+1})(\prod_{j=1}^{n} \omega_j)[\frac{1-J^{(n)}(t+s)}{1-J(t+s)}]}{\sum_{n=1}^{\infty} (1-\omega_{n+1})(\prod_{j=1}^{n} {}_j)[\frac{1-J^{(n)}(t)}{1-J(t)}]}\right\} . \tag{7.21}$$

By Lemmas A, B, and C in Chapter 3, we know that

$$\frac{1-J(t+s)}{1-J(t)} \to 1; \; \frac{1-J^{(n)}(t)}{1-J(t)} \to n;$$

and for any $\varepsilon > 0$ there is some $D(\varepsilon) < \infty$ making $\frac{1-J^{(n)}(t)}{1-J(t)} \le D(1+\varepsilon)^n$.

Thus if we choose ε to make $\omega(1-\varepsilon) < 1$,

$$\sum_{n=1}^{\infty} (1-\omega_{n+1})(\prod_{j=1}^{n} \omega_j)[\frac{1-J^{(n)}(t)}{1-J(t)}] \le D \sum_{n=1}^{\infty} (1-\omega_{n+1})[\omega(1-\varepsilon)]^n < \infty$$

and the Dominated Convergence Theorem implies

$$\lim_{t\to\infty} \sum_{n=1}^{\infty} (1-\omega_{n+1})(\prod_{j=1}^{n} \omega_j)[\frac{1-J^{(n)}(t)}{1-J(t)}] = \sum_{n=1}^{\infty} (1-\omega_{n+1})(\prod_{j=1}^{n} \omega_j)n. \tag{7.22}$$

Therefore

$$\lim_{t\to\infty} \frac{q(t+s)}{q(t)} = 1. \qquad \square$$

The next lemma generalizes Lemma 3.1.

Lemma 7.5

Suppose $\omega_n \le \omega < 1$ for all n and $J \in \mathcal{S}$. Then

$$\lim_{t\to\infty} P\{N(t)=k | A(t)=1\} = \frac{\prod_{j=1}^{k+1} \omega_j}{\sum_{n=0}^{\infty} (\prod_{j=1}^{n+1} \omega_j)} .$$

Proof:

When $J(x) \in \mathcal{S}$,

$$\frac{J^{(k)}(t)-J^{(k+1)}(t)}{J^{(n)}(t)-J^{(n+1)}(t)} = \left[\frac{J^{(k)}(t)-J^{(k+1)}(t)}{1-J(t)}\right]\left[\frac{1-J(t)}{J^{(n)}(t)-J^{(n+1)}(t)}\right]$$

But

$$\frac{J^{(r)}(t)-J^{(r+1)}(t)}{1-J(t)} = \frac{1-J^{(r+1)}(t)}{1-J(t)} - \frac{1-J^{(r)}(t)}{1-J(t)}$$

$$\to \{r + 1 - r\} = 1 \text{ as } t \to \infty \text{ for } r = 1, 2, \ldots$$

Thus

$$\frac{J^{(k)}(t)-J^{(k+1)}(t)}{J^{(n)}(t)-J^{(n+1)}(t)} \to 1 \text{ as } t \to \infty . \tag{7.23}$$

Now

$$P\{N(t)=k|A(t)=1\} = \frac{(\prod_{j=1}^{k+1} \omega_j)[J^{(k)}(t)-J^{(k+1)}(t)]}{\sum_{n=0}^{\infty} (\prod_{j=1}^{n+1} \omega_j)[J^{(n)}(t)-J^{(n+1)}(t)]}$$

$$= \frac{(\prod_{j=1}^{k+1} \omega_j)}{\sum_{n=0}^{\infty} (\prod_{j=1}^{n+1} \omega_j)[\frac{J^{(n)}(t)-J^{(n+1)}(t)}{J^{(k)}(t)-J^{(k+1)}(t)}]} \tag{7.24}$$

We know that

$$\frac{J^{(n)}(t)-J^{(n+1)}(t)}{J^{(k)}(t)-J^{(k+1)}(t)} < \left[\frac{1-J^{(n+1)}(t)}{1-J(t)}\right]\left[\frac{1-J(t)}{J^{(k)}(t)-J^{(k+1)}(t)}\right]$$

$$\leq D(1+\varepsilon)^{n+1} [1+o(1)]. \tag{7.25}$$

The Dominated Convergence Theorem applies to the denominator of (7.24) and the result follows. □

We will consider some of the consequences when $\prod_{j=1}^{\infty} \omega_j = \ell > 0$. As noted earlier, for this product to be nonzero, the ω_j's must approach one quickly enough to make $\sum_{j=1}^{\infty} (1-\omega_j) < \infty$. It is easy to see that when

$$\ell > 0, \lim_{t\to\infty} P\{N(t)=k|A(t)=1\} = 0 \text{ for all fixed } k.$$

Thus if the renewal process is alive at some time t and t is large, then with high probability the process has survived many lifetimes; intuition suggests the future development of the process will be much like that of a process whose lifetimes have proper distribution J.

Lemma 7.6

Suppose $\prod_{j=1}^{\infty} \omega_j = \ell > 0$. Then

$$E[N(t+s)-N(t)|A(t)=1] - [H_J(t+s)-H_J(t)] \to 0 \text{ as } t \to \infty.$$

Proof:

$$E[N(t+s)-N(t)|A(t)=1] = \frac{\sum_{n=1}^{\infty} (\prod_{j=1}^{n} \omega_j)[J^{(n)}(t+s)-J^{(n)}(t)]}{q(t)}. \tag{7.26}$$

Fix $\varepsilon > 0$. There exists an $N(\varepsilon)$ such that whenever $n \geq N$, $\prod_{j=1}^{n} \omega_j < \ell + \varepsilon$. Thus

$$E[N(t+s)-N(t)\,|\,A(t)=1]$$

$$\leq \frac{\sum_{n=1}^{N-1}[J^{(n)}(t+s)-J^{(n)}(t)]+(\ell+\varepsilon)\sum_{n=N}^{\infty}[J^{(n)}(t+s)-J^{(n)}(t)]}{\ell}$$

$$= o(1) + [H_J(t+s)-H_J(t)][1+\tfrac{\varepsilon}{\ell}] \; . \tag{7.27}$$

Similarly,

$$E[N(t+s)-N(t)\,|\,A(t)=1] \geq \frac{\ell[H_J(t+s)-H_J(t)]}{\ell + o(1)} \; . \tag{7.28}$$

Therefore the result follows. □

In order to study more general questions, suppose $G(t)$ is a natural process depending on the defective random variables X_1, $X_2,\cdots,X_{N(t)+1}$. Let $F_i(x) = \omega_i J(x)=P(X_i\leq x)$. Then

$$EG(t)A(t) = \sum_{n=0}^{\infty}\int_{\{N(t)=n\}} G(t,x_1,\cdots,x_{n+1})dF_1(x_1)\cdots dF_{n+1}(x_{n+1})$$

$$= \sum_{n=0}^{\infty}\int_{\{N(t)=n\}} G(t,x_1,\cdots,x_{n+1})\Big(\prod_{j=1}^{n+1}\omega_j\Big)dJ(x_1)\cdots dJ(x_{n+1})$$

$$= E_J G(t)\Big(\prod_{j=1}^{N(t)+1}\omega_j\Big) \tag{7.29}$$

where E_J indicates integration with respect to the proper distribution J.

Taking $G(t) \equiv 1$, we find

$$q(t) = E_J\left(\prod_{j=1}^{N(t)+1} \omega_j\right) \tag{7.30}$$

and

$$E[G(t)|A(t)=1] = \frac{E_J G(t)\left(\prod_{j=1}^{N(t)+1} \omega_j\right)}{E_J\left(\prod_{j=1}^{N(t)+1} \omega_j\right)} \quad . \tag{7.31}$$

Lemma 7.7

Suppose $|G(t)| \leq M$. If $\prod_{j=1}^{\infty} \omega_j = \ell > 0$, then

$E[G(t)|A(t) = 1] - E_J G(t) \to 0$ as $t \to \infty$.

Proof:

With no loss of generality, we may assume that $G(t) \geq 0$. Choose $\varepsilon > 0$. There exists an $N(\varepsilon)$ such that if $n \geq N$, $\left|\prod_{j=1}^{n+1} \omega_j - \ell\right| < \frac{\varepsilon}{M}$. Let $B_t = \{N(t) \geq N\}$.

By assumption,

$$E_J[G(t)(\prod_{j=1}^{N(t)+1} \omega_j)]$$

$$= E_J[G(t)(\prod_{j=1}^{N(t)+1} \omega_j)\chi(B_t)] + E_J[G(t)(\prod_{j=1}^{N(t)+1} \omega_j)\chi(B_t^c)]$$

$$= E_J[G(t)(\prod_{j=1}^{N(t)+1} \omega_j)\chi(B_t)] + o(1). \tag{7.32}$$

Also,

$$\ell E_J[G(t)\chi(B_t)] \le E_J[G(t)(\prod_{j=1}^{N(t)+1}\omega_j)\chi(B_t)] \le \ell E_J[G(t)\chi(B_t)] + \varepsilon \tag{7.33}$$

and

$$E_J(\prod_{j=1}^{N(t)+1}\omega_j) = \ell + o(1) \text{ as } t \to \infty. \tag{7.34}$$

Therefore,

$$E_J[G(t)]+o(1) \le \frac{E_J[G(t)(\prod_{j=1}^{N(t)+1}\omega_j)]}{E_J(\prod_{j=1}^{N(t)+1}\omega_j)} \le E_J[G(t)]+\varepsilon+o(1). \tag{7.35}$$

As ε is arbitrary, the lemma is proved. □

Applications of Lemma 7.7

Let $\mu_k(J) = \int_0^\infty x^k dJ(x)$. Suppose $W(t)$ is a cumulative process based upon $\{(X_i,Y_i)\}$ where $X_i \sim \omega_i J$ and $E_J Y^r = \kappa_r(J)$. If $\kappa_1^*(J) < \infty$ and $\mu_1(J) < \infty$, then let

$$G(t) = \chi(\ |\frac{W(t)}{t} - \frac{\kappa_1(J)}{\mu_1(J)}| > \varepsilon).$$

If $\prod_{j=1}^{\infty}\omega_j = \ell > 0$, it follows that

$$P\{|\frac{W(t)}{t} - \frac{\kappa_1(J)}{\mu_1(J)}| > \varepsilon | A(t)=1\} - P_J\{|\frac{W(t)}{t} - \frac{\kappa_1(J)}{\mu_1(J)}| > \varepsilon\} \to 0$$

and hence

$$P\{|\frac{W(t)}{t} - \frac{\kappa_1(J)}{\mu_1(J)}| > \varepsilon | A(t) = 1\} \to 0 \text{ as } t \to \infty. \qquad (7.36)$$

If $\kappa_2^*(J) < \infty$ and $\mu_2(J) < \infty$, define

$$G(t) = \chi\left(\frac{W(t) - \frac{\kappa_1(J)}{\mu_1(J)}t}{\sqrt{\frac{\gamma_J t}{\mu_1(J)}}} \le \alpha\right)$$

where $\gamma_J = E_J[Y_1 - \frac{\kappa_1(J)}{\mu_1(J)} X_1]^2$. Lemma 7.7 implies

$$P\left\{\frac{W(t) - \frac{\kappa_1(J)}{\mu_1(J)}t}{\sqrt{\frac{\gamma_J t}{\mu_1(J)}}} \le \alpha \,\middle|\, A(t)=1\right\} \to \Phi(\alpha) \text{ as } t \to \infty \qquad (7.37)$$

whenever $\prod_{j=1}^{\infty} \omega_j = \ell > 0$.

Lemma 7.7 only applies to bounded natural processes. The next lemma gives a crude indication of the behavior of more general processes.

Lemma 7.8

Suppose $G(t)$ is a natural process such that $E_J G(t)\chi(N(t) \le N) \to 0$ for every fixed N. If, in addition,

$$\prod_{j=1}^{\infty} \omega_j = \ell > 0,$$

then

$$E[G(t)|A(t) = 1] \sim E_J G(t).$$

Proof:

Again we may assume $G(t) \geq 0$. Choose $\varepsilon > 0$. There exists some $N(\varepsilon)$ such that $\left|\prod_{j=1}^{n+1} \omega_j - \ell\right| < \varepsilon$ whenever $n \geq N$. Let $B_t = \{N(t) \geq N\}$.

$$E_J[G(t)(\prod_{j=1}^{N(t)+1} \omega_j)] = E_J[G(t)(\prod_{j=1}^{N(t)+1} \omega_j)\chi(B_t)]$$

$$+ E_J[G(t)(\prod_{j=1}^{N(t)+1} \omega_j)\chi(B_t^c)].$$

Again

$$E_J[G(t)(\prod_{t=1}^{N(t)+1} \omega_j)\chi(B_t^c)] \leq E_J G(t)\chi(B_t^c) = o(1)$$

by assumption.

$$\ell E_J[G(t)\chi(B_t)] \leq E_J[G(t)(\prod_{j=1}^{N(t)+1} \omega_j)\chi(B_t)] \leq (\ell+\varepsilon)E_J[G(t)\chi(B_t)]$$

and thus

$$\ell E_J[G(t)] + o(1) \leq E_J[G(t)(\prod_{j=1}^{N(t)+1} \omega_j)] \leq (\ell+\varepsilon)E_J[G(t)]+o(1).$$

The result follows. □

Application of Lemma 7.8

Suppose $G(t) = N(t)^k$. If $\prod_{j=1}^{\infty} \omega_j = \ell > 0$, then Lemma 7.8 implies

$$E[N(t)^k | A(t) = 1] \sim E_J N(t)^k. \tag{7.38}$$

If $\mu_{k+1}(J) < \infty$, this indicates

$$E[N(t)^k | A(t) = 1] = ct^k + o(t^k).$$

Of course, as we indicated earlier, the assumption that $\prod_{j=1}^{\infty} \omega_j = \ell > 0$ is a very strong one. Rather than assume $\omega_j \to 1$ with a certain speed, it may be more reasonable to suppose that $\omega_j \to \omega < 1$. As before, let $F_j(x) = \omega_j J(x) = P\{X_j \leq x\}$. If there exists a $\sigma > 0$ making

$$\omega \int_0^\infty e^{\sigma x} dJ(x) = 1 \tag{7.39}$$

then

$$d\tilde{F}_j(x) = \frac{\omega}{\omega_j} e^{\sigma x} dF_j(x) = \omega e^{\sigma x} dJ(x) \equiv d\tilde{F}(x) \tag{7.40}$$

defines a proper distribution $\tilde{F}(x)$.

Let $G(t)$ be natural and let $\tilde{E}$ indicate integration with respect to the proper distribution $\tilde{F}(x)$. Then

$$\tilde{E}G(t) = \sum_{n=0}^{\infty} \int_{\{N(t)=n\}} G(t, x_1, \cdots, x_{n+1}) d\tilde{F}(x_1) \cdots d\tilde{F}(x_{n+1})$$

$$= \sum_{n=0}^{\infty} \int_{\{N(t)=n\}} G(t,x_1,\cdots,x_{n+1}) \left(\prod_{j=1}^{n+1} \frac{\omega}{\omega_j} \right) e^{\sigma t} e^{\sigma \zeta_t} dF_1(x_1) \cdots dF_{n+1}(x_{n+1})$$

and thus

$$e^{-\sigma t} \tilde{E} G(t) e^{-\sigma \zeta_t} \left(\prod_{j=1}^{N(t)+1} \frac{\omega_j}{\omega} \right) = EG(t)A(t) \tag{7.41}$$

$$q(t) = e^{-\sigma t} \tilde{E} e^{-\sigma \zeta_t} \left(\prod_{j=1}^{N(t)+1} \frac{\omega_j}{\omega} \right). \tag{7.42}$$

Therefore

$$E[G(t) | A(t)=1] = \frac{\tilde{E} G(t) e^{-\sigma \zeta_t} \left(\prod_{j=1}^{N(t)+1} \frac{\omega_j}{\omega} \right)}{\tilde{E} e^{-\sigma \zeta_t} \left(\prod_{j=1}^{N(t)+1} \frac{\omega_j}{\omega} \right)} . \tag{7.43}$$

<u>Lemma 7.9</u>

Suppose $\prod_{j=1}^{\infty} \frac{\omega_j}{\omega} = \ell > 0$. If there is a σ making $d\tilde{F}(x) = \omega e^{\sigma x} dJ(x)$ a proper distribution, and if $\tilde{\mu}_1 = \int_0^{\infty} x d\tilde{F}(x) < \infty$, then

$$\tilde{E} e^{-\sigma \zeta_t} \left(\prod_{j=1}^{N(t)+1} \frac{\omega_j}{\omega} \right) \to \ell \left[\frac{1-\omega}{\sigma \tilde{\mu}_1} \right] = \ell \tilde{K}^*(\sigma) \text{ as } t \to \infty.$$

Proof:

$$\tilde{E} e^{-\sigma \zeta_t} \left(\prod_{j=1}^{N(t)+1} \frac{\omega_j}{\omega} \right) = \tag{7.44}$$

$$\frac{\omega_1}{\omega} \int_t^{\infty} e^{-\sigma(x-t)} d\tilde{F}(x) + \sum_{n=1}^{\infty} \left(\prod_{j=1}^{n+1} \frac{\omega_1}{\omega} \right) \int_0^t \int_{t-\tau}^{\infty} e^{-\sigma(y+\tau-t)} d\tilde{F}(y) d\tilde{F}^{(n)}(\tau).$$

Choose $\varepsilon > 0$. There exists an $N(\varepsilon)$ such that

$$\left|\prod_{j=1}^{n+1} \frac{\omega_j}{\omega} - \ell\right| < \varepsilon \qquad \text{whenever } n \geq N.$$

Then

$$\tilde{E}e^{-\sigma\zeta_t}\left(\prod_{j=1}^{N(t)+1} \frac{\omega_j}{\omega}\right)$$

$$< \frac{\omega_1}{\omega}\int_t^\infty e^{-\sigma(x-t)}d\tilde{F}(x) + \sum_{n=1}^{N-1}\left(\prod_{j=1}^{n+1}\frac{\omega_j}{\omega}\right)\int_0^t\int_{t-\tau}^\infty e^{-\sigma(y+\tau-t)}d\tilde{F}(y)d\tilde{F}^{(n)}(\tau)$$

$$+ (\ell+\varepsilon)\sum_{n=1}^{\infty}\int_0^t\int_{t-\tau}^\infty e^{-\sigma(y+\tau-t)}d\tilde{F}(y)d\tilde{F}^{(n)}(\tau). \qquad (7.45)$$

$$\lim_{t\to\infty} \tilde{E}e^{-\sigma\zeta_t}\left(\prod_{j=1}^{N(t)+1} \frac{\omega_j}{\omega}\right) \leq \lim_{t\to\infty} \frac{\omega_1}{\omega}\,[1-\tilde{F}(t)]$$

$$+ \lim_{t\to\infty}\sum_{n=1}^{N-1}\left(\prod_{j=1}^{n+1}\frac{\omega_j}{\omega}\right)[\tilde{F}^{(n)}(t)-\tilde{F}^{(n+1)}(t)]$$

$$+ (\ell+\varepsilon)\lim_{t\to\infty}\int_0^t\int_{t-\tau}^\infty e^{-\sigma(y+\tau-t)}d\tilde{F}(y)d\tilde{H}(\tau)$$

$$= \frac{(\ell+\varepsilon)(1-\omega)}{\sigma\tilde{\mu}_1} \quad \text{by the Key Renewal Theorem.} \qquad (7.46)$$

Similarly, one can show

$$\lim_{t\to\infty} \tilde{E}e^{-\sigma\zeta_t}\left(\prod_{j=1}^{N(t)+1} \frac{\omega_j}{\omega}\right) \geq \frac{\ell(1-\varepsilon)}{\sigma\tilde{\mu}_1}.$$

This proves the result. $\square$

Lemma 7.10

Assume the proper distribution $\tilde{F}$ exists. Also suppose G(t) is a sluggish and natural process such that for every $\varepsilon > 0$ there exists some $\Delta(\varepsilon)$ making

$$\tilde{E}G(t)\chi(|G(t)| > \Delta) < \varepsilon$$

for all sufficiently large t. Then if $\prod_{j=1}^{\infty} \frac{\omega_j}{\omega} = \ell > 0$,

$$E[G(t)|A(t)=1] - \tilde{E}G(t) \to 0 \text{ as } t \to \infty.$$

Proof:

Fix $\varepsilon > 0$. There exists some $N(\varepsilon)$ making $\left|\prod_{j=1}^{n+1} \frac{\omega_j}{\omega} - \ell\right| < \varepsilon$ whenever $n \geq N$. Let $B_t = \{N(t) \geq N\}$.

$$\tilde{E}e^{-\sigma\zeta_t}G(t)\left(\prod_{j=1}^{N(t)+1} \frac{\omega_j}{\omega}\right) = \tilde{E}e^{-\sigma\zeta_t}G(t)\chi(B_t)\left(\prod_{j=1}^{N(t)+1} \frac{\omega_j}{\omega}\right)$$

$$+ \tilde{E}e^{-\sigma\zeta_t}G(t)\chi(B_t^c)\left(\prod_{j=1}^{N(t)+1} \frac{\omega_j}{\omega}\right). \tag{7.47}$$

But

$$\left|\tilde{E}e^{-\sigma\zeta_t}G(t)\chi(B_t^c)\left(\prod_{j=1}^{N(t)+1} \frac{\omega_j}{\omega}\right)\right| \leq M|\tilde{E}G(t)\chi(B_t^c)|$$

where M is an upper bound on $\prod_{j=1}^{m} \frac{\omega_j}{\omega}$ for $m=1,\cdots,N$.

$$\leq M\tilde{E}G(t)\chi(|G(t)| > \Delta) + M\Delta\tilde{P}(B_t^c) < \varepsilon \tag{7.48}$$

if we choose Δ correctly and if t is sufficiently large.

Also,

$$\tilde{E}e^{-\sigma\zeta_t}G(t)\chi(B_t) \le \tilde{E}e^{-\sigma\zeta_t}G(t)\chi(B_t)\left(\prod_{j=1}^{N(t)+1}\frac{\omega_j}{\omega}\right)$$

$$\le (\ell+\varepsilon)\tilde{E}e^{-\sigma\zeta_t}G(t)\chi(B_t). \tag{7.49}$$

Therefore

$$-\varepsilon+\ell\tilde{E}e^{-\sigma\zeta_t}G(t) \le \tilde{E}e^{-\sigma\zeta_t}G(t)\left(\prod_{j=1}^{N(t)+1}\frac{\omega_j}{\omega}\right) \le (\ell+\varepsilon)\tilde{E}e^{-\sigma\zeta_t}G(t)+\varepsilon \tag{7.50}$$

for sufficiently large t. Hence

$$E[G(t)|A(t)=1] - \frac{\ell\tilde{E}e^{-\sigma\zeta_t}G(t)}{\ell\left[\frac{1-\omega}{\sigma\tilde{\mu}_1}\right]} \to 0 \tag{7.51}$$

by Lemma 7.9 and expression (7.50). Now Lemma 4.5 implies

$$E[G(t)|A(t)=1] - \tilde{E}G(t) \to 0. \qquad \square$$

This lemma allows us to extend several of the results in Chapter 4 to the situation in which the ω_j's are not necessarily identical but do converge to some ω quickly enough to make $\prod_{j=1}^{\infty}\frac{\omega_j}{\omega} = \ell$. For example, by linking Lemma 7.10 and Theorem 4.1, we find that if $\tilde{\mu}_1 < \infty$ and $\tilde{\kappa}_2^* < \infty$, then

$$\text{(i)}\quad P\left\{\frac{W(t)-\tilde{\kappa}_1 N(t)}{\tilde{\sigma}_y\sqrt{\frac{t}{\tilde{\mu}_1}}} \le \alpha \,\middle|\, A(t) = 1\right\} \to \Phi(\alpha) \tag{7.52}$$

and if $\tilde{\mu}_2 < \infty$

$$\text{(ii)}\quad P\left\{\frac{W(t)-\dfrac{\tilde{\kappa}_1 t}{\tilde{\mu}_1}}{\sqrt{\dfrac{\tilde{\gamma}t}{\tilde{\mu}_1}}} \le \alpha \;\middle|\; A(t)=1\right\} \to \Phi(\alpha). \tag{7.53}$$

Linking Theorem 4.2 and Lemma 7.10 yields

$$E[(W(t) - \frac{\tilde{\kappa}_1 t}{\tilde{\mu}_1})^2 \,|A(t) = 1] = \frac{\tilde{\gamma}_t}{\tilde{\mu}_1} + o(t) \tag{7.54}$$

if $\tilde{\mu}_2 < \infty$ and $\tilde{\kappa}_2^* < \infty$.

Thus we see that the assumption that all lifetimes have the same defect ω can be relaxed somewhat without overturning our results.

APPENDIX

Theorem G

Suppose that q is a positive integer, $m \geq 0$, and

1) $F \in C$ and $\tilde{\mu}_{m+1} < \infty$.

2) $J(t)$ is a function of bounded variation on $[0,\infty)$ having $m + q$ absolute moments and $J^*(s) = 0(|s|^q)$ as $|s| \to 0$.

3) $\Psi^*(s)$ is defined for $R(s) > 0$ by

$$\Psi^*(s) = \frac{J^*(s)}{[1-\tilde{F}^*(s)]^q}$$

and $\Psi^*(0)$ is defined to make $\Psi^*(s)$ continuous.

Then $\Psi(t) \in B(m)$, the class of functions of bounded variation on $[0,\infty)$ having m absolute moments.

Proof:

This theorem is a special case of Theorem 1 in Smith (1966). Smith's Theorem is couched in terms of Fourier-Stieltjes transforms but can be adapted to the current Laplace-Stieltjes setting. In this instance we take $\alpha_1 = \gamma = q$ and $M(x) = x^m$.

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